Matching with Quotas

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July 31, 2016

Abstract

The paper characterizes the core of many-to-one matching problem with quotas. The many-to-one matching problem with quotas may have an empty core, and there is no clear set of necessary and sufficient conditions that guarantee non-emptiness of the core. Usual sufficient conditions for non-emptiness of the core for matching problems cannot be applied for the problem with quotas. We introduce set strong substitutability of preferences, a refinement of strong substitutability for the problem with quotas. We show that if preferences are set strongly substitutable, then the core of many-to-one matching problem with quotas is non-empty. Moreover, we prove that in this case the core has a lattice structure with opposition of interests.

Keywords: Matching, Stability, Fixed Point, Quotas

JEL classification: D62; C78

1 Introduction

The matching problem was introduced by Gale and Shapley (1962). Gale and Shapley also proposed a mechanism for finding a stable solution in one-to-one and many-to-one cases. Many-to-one matching problem is well-studied and has various applications. For instance, National Residency Matching Program (Roth and Peranson (1999)), Boston and New York public school matching procedures (Abdulkadiroğlu et al. (2005b) and Abdulkadiroğlu et al. (2005a)), United States Military Academy cadets assignments (Sönmez and Switzer (2013)), German, Hungarian, Spanish and Turkish College Admission mechanisms (Braun et al.,Biró

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We are grateful to Marco Castillo, David Eil and César Martinelli for their helpful comments.

(2008), Romero-Medina (1998) and Balinski and Sönmez (1999)), Japan Residency Matching Program (Kamada and Kojima (2011)).

Most of many-to-one matching problems listed above come with capacity restrictions. For instance, rural hospitals require a minimum of doctors to operate and can not employ more than a certain maximum amount of doctors. Indeed, one can think of the use of capacity constraints as a way to secure the assignment of doctors across locales. Despite its clear practical implications, we know little about the core in many-to-one problems with quotas. Many-to-one matching problem is different from the one-to-one matching problem in various dimensions. The core of many-to-one matching problem can be empty unlike the one for one-to-one problem. There are several sufficient conditions for non-emptiness of the core of many-to-one matching problem (Roth (1985a) and Blair (1988)). These results hold for the problem with upper quotas (capacities), while can not be directly applied for the problem with lower quotas. Moreover, to our knowledge there is no sufficient condition for non-emptiness of the core of many-to-one matching problem with quotas. We study the case in which these restrictions can not be violated as in Biró et al. (2010).[1] That is, when each assignment is required to satisfy both minimum and maximum amount of matches.

We provide a condition on preferences that secures the existence of a matching in the core of the problem with quotas. Moreover, we show that under the assumption, the core is a lattice with the opposition of the interests.[2] We prove the results using the fixed point approach from Echenique and Oviedo (2004) and Echenique and Oviedo (2006).

Hatfield and Milgrom (2005) show that substitutability[3] of preferences guarantees non-emptiness of the core for the matching with contracts. The importance of the substitutability condition for non-emptiness of the core has been shown for supply chain networks (Ostrovsky (2008)), ascending clock auctions (Milgrom and Strulovici (2009)), package auctions (Milgrom (2007)), many-to-one and many-to-many matching problems (Echenique and Oviedo (2004); Echenique and Oviedo (2006)). We show that substitutability of preferences is not a feasible assumption for the problem with quotas. That is preferences over possible assignments in the problem with quotas can not satisfy substitutability.[4]

The remainder of this paper is organized as follows. In Section 2 we state the model of many-to-one matching with quotas. In Section 3 we show the results on non-emptiness of the core and the lattice structure of it. In Section 4 we discuss connection of the results to the previous literature and show that the conditions for non-emptiness of the core which were obtained before can not be directly applied for the problem with quotas.

[1] For the case in which these restrictions can be violated see Fragiadakis et al. (2016).

[2] Under opposition of interests we mean conflicting interests as it was introduced by Roth (1985b).

[3] Substitutability of preferences informally imply that all commodities or partners are substitutes. For the formal definition of substitutability see Section 4. For the expansive discussion on substitutability see Hatfield et al. (2016)

[4] This follows from the formal definition of the substitutability. For the formal statement see Lemma 5.

2 Preliminaries

A matching problem can be specified as a tuple $\Gamma = (N, \mathcal{M}, \mathcal{R})$, where N is the set of players, $\mathcal{M}$ is the set of all possible matchings and $\mathcal{R}$ is the preference profile.

Let us start from defining N, the set of players. For simplicity, we will call the two participating sides as students and courses. We use $S = \{s_1, ..., s_n\}$ to denote the set of students, and $C = \{c_1, ..., c_m\}$ to denote the set of courses, and $N = S \cup C$.

2.1 Matching

Recall that we consider a matching problem with quotas. Therefore, to define $\mathcal{M}$ we need to define quotas first. Let the lower quotas be a function $\underline{q}(c) : C \to \mathbb{N}$ and for any agent $c \in C$, an upper quotas be a function $\bar{q}(c) : C \to \mathbb{N}$ such that $\bar{q}(c) \geq \underline{q}(c)$. Every course has to get matched with at least $\underline{q}(c)$ partners and no more than $\bar{q}(c)$. Note that since we are considering many-to-one matching problem, students can be matched to either one or none of the courses.

An assignment is a correspondence $\nu = (\nu_S, \nu_C)$, where $\nu_S : S \to C \cup \{\emptyset\}$ and $\nu_C : C \to 2^S$. A **prematching** is an assignment such that $s \in \nu_C(c)$ if and only if $c = \nu_S(s)$. Recall that students can be matched to no more than one course, while courses can be matched to many students. For the simplicity of further notation we can refer to the match of agent a as $\nu(a)$ that would be equal to $\nu_C(a)$ if $a \in C$ or $\nu_S(a)$ if $a \in S$.

Definition 1. *A prematching μ is said to be a* ***matching*** *if there are $S' \subseteq S$ and $C' = \bigcup_{s \in S'} \mu_S(s) \subseteq C$, such that*

(i) for every $c \in C'$, $\underline{q}(c) \leq |\mu(c)| \leq \bar{q}(c)$ and,

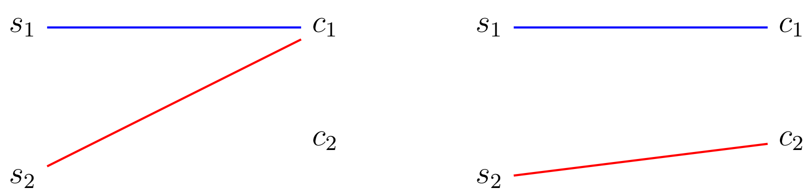
(ii) for every $\bar{c} \in C \setminus C'$ $\mu_C(\bar{c}) = \emptyset$ and,

(iii) for every $\bar{s} \in S \setminus S'$ $\mu_S(\bar{s}) = \emptyset$

Note that presence of quotas requires that course is either unmatched or matched to at least $\underline{q}(c)$ students. Let $q = (\underline{q}, \bar{q})$ be vector of lower and upper quotas for all players, then we can denote by $\mathcal{M}(C, S, q)$ set of all matchings. The set can be characterized by the set of players (students and courses) and the quotas.

Example: Let $C = \{c_1, c_2\}$, $S = \{s_1, s_2\}$ and $\underline{q}(c_1) = \underline{q}(c_2) = 2$ and $\bar{q}(c_1) = \bar{q}(c_2) = 3$

Figure 1 illustrates the definition of Matching. Figure 1(a) shows the example of matching, where c_2 is unmatched, but s_1 and s_2 are matched with c_1. Therefore, c_1 fulfills its lower quota. Figure 1(b) shows example of prematching which is not a matching, both students are matched as well as both courses, but neither course fulfills the lower quota.



(a) Matching (b) Prematching, but not a matching

Figure 1: Illustration of difference between matching and prematching

2.2 Preferences

Now we need to define the preference profile $\mathcal{R}$, the set of preference relations. We assume that every agent $a \in C \cup S$ has a linear order[5] of preferences over possible sets of partners (possible matches). Denote by $R(s)$ the preference relation of student s and by $\tilde{R}(c)$ the preference relation of course c. We use $\tilde{R}(c)$ since we will have to change the preference profile to incorporate the quotas in it lates, while $R(s)$ would remain unchanged.

A set of partners (match) X is said to be **acceptable** by agent a if $X R(a) \emptyset$. Let $R(c)$ be the *truncated preference relation*, such that any $X \neq \{\emptyset\}$ with $|X| < \underline{q}(a)$ or $|X| > \bar{q}(c)$ is unacceptable. Note that if $\tilde{R}(c)$ is a linear order, then $R(c)$ is a linear order as well, since it is a permutation of $\tilde{R}(c)$ that makes several sets of students unacceptable. Denote by $P(a)$ the strict part of preference relation of $R(a)$, for any $a \in C \cup S$.

Example: Let $C = \{c_1; c_2\}$, $S = \{s_1, s_2, s_3\}$ and $\underline{q}(c_1) = \underline{q}(c_2) = 2$ and $\bar{q}(c_1) = \bar{q}(c_2) = 2$.

$$\tilde{R}(c_1) : \{s_1, s_2, s_3\} P(c_1) \{s_1, s_2\} P(c_1) \{s_1, s_3\} P(c_1) s_1 P(c_1) \emptyset \qquad R(c_1) : \{s_1, s_2\} P(c_1) \{s_1, s_3\} P(c_1) \emptyset$$

(a) Original preference relation (b) Truncated preference relation

Figure 2: Illustration of truncated preference relation for courses

Figure 2 illustrates the construction of truncated preference relation from the original one. Figure 2(a) shows the original preference relation over sets of alternatives. In this case the set $\{s_1, s_2, s_3\}$ is unacceptable, since it exceeds the capacity (upper quota) and the alternative s_1 is unacceptable since it would not allow to fulfill lower quota. Eliminating these alternatives we arrive to the truncated preference relation shown in Figure 2(b).

Let $Ch(X, R(a)) \subseteq X$ be the most preferred set of agent a.[6] Note that since we assume $R(a)$ to be a linear order, $Ch(X, R(a))$ is unique. So, $Ch(X, R(a))$ is the unique subset X' of X, such that $X' P(a) X''$ for any $X'' \subseteq X$. Note that for any $c \in C$ and $X \subseteq S$ if $|X| \leq \underline{q}(c)$ or $|X| \geq \bar{q}(c)$, then $Ch(X, R(c)) = \{\emptyset\}$.[7] Let us characterize some properties of a choice

[5] A linear order is a complete, anti-symmetric and transitive preference relation

[6] In this case X is an arbitrary set that includes $\emptyset$ as an element.

[7] This statement is correct since we use the truncated preference relation of course $c \in C$.

function generated by a linear order which we will use later:

- The choice function is **idempotent**. That is $Ch(Ch(X, R(a)), R(a)) = Ch(X, R(a))$.
- The choice function is **monotone**. That is for any $X \subseteq X'$, $Ch(X', R(a))\ R(a)\ Ch(X, R(a))$.

Definition 2. *An agent a's preference $R(a)$ satisfies **set strong substitutability** if for any X, X' such that $X'R(a)X$ and S', such that $|X \cup S'| \in \{0\} \cup [\underline{q}(a); \bar{q}(a)]$ and $|X' \cup S'| \in \{0\} \cup [\underline{q}(a); \bar{q}(a)]$:*

$$x \in S' \text{ and } x \in Ch(X' \cup S', R(a)) \;\Rightarrow\; x \in Ch(X \cup S', R(a))$$

Let us consider an example to illustrate what set strong substitutability is. Assume that there are three students $s_1, s_2, s_3 \in S$. If $X' = \{s_1, s_2\}R(c)\{s_1\} = X$, $S' = s_3$ and $Ch(X' \cup S, R(c)) = \{s_2, s_3\}$, then, $\{s_1\} = Ch(X \cup S', R(c))$ violates set strong substitutability. It implies that s_2 and s_3 are complementary, i.e. their combination overweights the combination of s_1 and s_3 as well as single s_1, because s_3 is not chosen in the absence of s_2.

A preference profile $\mathcal{R}$ is **set strongly substitutable** if $R(c)$ satisfies set strong substitutability for every course $c \in C$. Note that for $\mathcal{R}$ to be set strongly substitutable we require only preferences of courses to satisfy set strong substitutability. Since for any $s \in S$ $R(S)$ is a linear order over $C \cup \{\emptyset\}$, then it would satisfy set strong substitutability. Note that requiring $\mathcal{R}$ to be set strongly subsistitutable is cruicial for the proof of our main result.

2.3 Solution Concept

We define the solution concept for matching problem Γ as follows. A matching μ is said to be **individually rational** if $\mu(a) = Ch(\mu(a), R(a))$ for any $a \in C \cup S$. We consider the core as the solution concept since it is a general concept that can be defined for a game in abstract form.

Let us start from defining the core.[8] For this purpose we need to introduce the dominance relation.

Definition 3. *A matching ν **dominates** a matching μ if there is a $S' \neq \emptyset$, $S' \subseteq S$ and $C' = \bigcup_{s_i \in S'} \nu_S(s_i) \subseteq C$, such that*

(i) for every $a \in C' \cup S'$ $\nu(a)R(a)\mu(a)$, and

(ii) for some $a \in C' \cup S'$ $\nu(a)P(a)\mu(a)$.

[8] The definition of the core we are using is sometimes called an individually rational core, since it constraints the set of dominant matchings to the individually rational matchings.

Note that the dominance relation requires ν to be matching as well, i.e. the prematching that satisfies quotas. Dominance (or blocking) as it is defined in the literature on matching without quotas allows a prematching to dominate a matching. We restrict the set of feasible coalitions to matchings only, therefore matchings cannot be dominated by prematchings that are not matchings. That is if a course wants to leave the market and get better students on its own, it still has to fulfil lower quotas.

Definition 4. ***Core*** *($\mathcal{C} \subseteq \mathcal{M}$) is a collection of individually rational matchings $\mu \in \mathcal{C}$, such that there is no individually rational matching $\mu' \in \mathcal{M}$ that dominates μ.*

Note that if μ' dominates individually rational μ it does not necessarily imply the individual rationality of μ', since it may be the case that μ' assigns some unacceptable partners to some agents. However, if there is μ' that dominates individually rational μ then there always exists an individually rational μ'' that dominates μ. For instance, we can take all agents that are assigned unacceptable partners and assign them $\emptyset$, since these people are not in $C' \cup S'$. Therefore, changing partners assigned to these agents would not affect dominance.

3 Results

Let us now state the main result of the paper: a sufficient condition for non-emptyness of the core.

Theorem 1. *If the preference profile $\mathcal{R}$ is set strongly substitutable, then the core is not empty.*

The proof has the following structure. We introduce an operator T and show that the set of its fixed points is a subset of the core of Γ. Second, we prove that T is a monotone operator over a complete lattice. This will allow us to apply Tarski's fixed point theorem to show that $\mathcal{E}$ is non-empty, and therefore the core is non-empty.

Therfore, proof of Theorem 1 requires the following Lemmata.

3.1 Lemmata

To prove Theorem 1 we need to introduce the following definitions.

Denote by $\mathcal{V}_S = (C \cup \{\emptyset\})^S$ the set of all assignments that students can get. Denote by $\mathcal{V}_C \subseteq (2^S)^C$ set of all assignments that satisfies courses' quotas. Say that a pair $\nu = (\nu_S, \nu_C)$ with $\nu_S : S \to C \cup \{\emptyset\}$ and $\nu_C : C \to 2^S$ and $\nu_S \in \mathcal{V}_S$, $\nu_C \in \mathcal{V}_C$ is an **assignment**. Note that difference between matching and assignment is that an assignment does not have to be mutually consistent, i.e. if $s \in \nu_C(c)$ in assignment ν, it does not require that $c = \nu_S(s)$. Let

$\mathcal{V} = \mathcal{V}_S \times \mathcal{V}_C$. Let us define the following partial orders over $\mathcal{V}$. For any assignment $\nu \in \mathcal{V}$ of agent a, either $\nu(a) = \emptyset$, or $\underline{q}(a) \leq |\nu(a)| \leq \bar{q}(a)$. Therefore, the required preference profile is defined only over the assignments that fill out quotas and the assignment to "stay alone", that is, assignment to the empty set.

Now we move on to introducing the T operator.

Definition 5. *Let ν be an assignment, then*

- *Let $U(c,\nu) = \{s \in S : \ cR(s)\nu(s)\}$ for any $c \in C$*
- *Let $V(s,\nu) = \{c \in C : \exists S' \subseteq S : \ s \in S' \cap Ch(\nu(c) \cup S', R(c))\}$ for any $s \in S$.*

Set $U(c,\nu)$ is the set of students that prefer c to the current assignment of s. Set $V(s,\nu)$ is the set of courses that would include s into their most preferred set from $\nu_C(c) \cup S'$ for some S' that includes s.

Definition 6. *Now define $T : \mathcal{V} \to \mathcal{V}$ by*

$$(Tv)(a) = \begin{cases} Ch(U(a,\nu), R(a)) \ if \ a \in C \\ Ch(V(a,\nu), R(a)) \ if \ a \in S \end{cases}$$

for any $a \in C \cup S$.

Then T operator assigns to every $c \in C$ the most preferred set of students among those $s \in S$ that prefers c to the current assignment and to every $s \in S$ the most preferred course among those $c \in C$ that chooses s from some $\nu_C(c) \cup S'$, such that S' contains s. An assignment ν is said to be a **fixed point** of T if $T\nu = \nu$. Denote by $\mathcal{E}$ the set of fixed points, i.e. $\mathcal{E} = \{\nu \in \mathcal{V} : \ \nu = T\nu\}$.

Hence, the **T-algorithm** is the procedure of iterating T starting at some assignment $\nu \in \mathcal{V}$. Note that operator T starts from an assignment and iterates by creating a new assignment that is not necessarily a prematching or a matching, but satisfies quotas. Hence, we need to show that if T-algorithm stops, then it stops at the matching.[9]

Lemma 1. *If $\nu \in \mathcal{E}$, then ν is an individually rational matching*

Proof. Let $\nu = (\nu_C, \nu_S) \in \mathcal{E}$. We first show that ν is a prematching, and then we show that it respects quotas - that is ν is a matching. Recall that ν is an assignment since it is element of $\mathcal{V}$. Hence we need to show that $s \in \nu_C(c)$ if and only if $\nu_S(s) = c$.

$(s \in \nu_C(c) \ \Rightarrow \ \nu_S(s) = c)$

Fix $s \in \nu_C(c)$. Then $\nu \in \mathcal{E}$ implies that $\nu(c) = (T\nu)(c) = Ch(U(c,\nu), R(c)) \ \Rightarrow \ s \in$

[9] Since T-algorithm can stop only at the fixed point of T operator.

$U(c,\nu)$. By definition of $U(c,\nu)$: $cR(s)\nu(s)$. Then $Ch(\nu_C(c), R(c)) = Ch((T\nu)(c), R(c)) = Ch(Ch(U(c,\nu), R(c)), R(c))$. Recall that choice function is idempotent, hence, $Ch(Ch(U(c,\nu), R(c)), R(c)) = Ch(U(c,\nu), R(c))$ and $Ch(U(c,\nu), R(c)) = \nu_C(c)$. Then, $Ch(U(c,\nu), R(c)) = \nu_C(c)$. Hence, $Ch(\nu_C(c), R(c)) = \nu_C(c)$, i.e. $\nu_C(c)$ is individually rational.

Since, $s \in \nu_C(c)$, then $Ch(\nu_C(c), R(c)) = Ch(\nu_C(c) \cup \{s\}, R(c))$. Then $c \in V(s,\nu)$. At the same time $\nu_S(s) = (T\nu)(s) = Ch(V(s,\nu), R(s))$, therefore, $\nu_S(s) \subseteq V(s,\nu)$. Then, $Ch(\nu_S(s)\cup\{c\}, R(s)) \subseteq \nu_S(s)\cup\{c\} \subseteq V(s,\nu)$. Since $\nu_S(s)$ is chosen from, $V(s,\nu)$ $\nu_S(s)R(s)Ch(\nu_S(s)\cup \{c\}, R(s))$, hence $\nu_S(s)R(s)cR(s)\nu_S(s)$. Therefore, $\nu_S(s) = c$.

$(c = \nu_S(s) \;\Rightarrow\; s \in \nu_C(c))$

Fix $c = \nu_S(s)$. Then $\nu \in \mathcal{E}$ implies $\nu(s) = (T\nu)(s) = Ch(V(s,\nu), R(s))$. Hence, there is $S' \subseteq S$, such that $s \in Ch(\nu_C(c) \cup S', R)$. $Ch(\nu_S(s), R(s)) = Ch((T\nu)(s), R(s)) = Ch(Ch(V(s,\nu), R(s)), R(s))$. Recall that choice function is idempotent, hence, $Ch(\nu_S(s), R(s)) = Ch(Ch(V(s,\nu), R(s)), R(s))$ and $Ch(V(s,\nu), R(s)) = \nu_S(s)$. Then, $Ch(V(s,\nu), R(s)) = \nu_S(s)$. Hence, $Ch(\nu_S(s), R(s)) = \nu_S(s)$, i.e. $\nu_S(s)$ is individually rational

Since $c = \nu_S(s)$, then $s \in U(c,\nu)$ and $\nu_C(c) \subseteq U(c,\nu)$ since ν is a fixed point. But $\nu_C(c) = (T\nu_C)(c) = Ch(U(c,\nu_C(c)), R(c))$ then $\nu_C(c) \subseteq U(c,\nu_C(c))$. Then, $Ch(\nu_C(c) \cup \{s\}, R(c)) \subseteq \nu_C(c) \cup \{s\} \subseteq U$. Hence, $Ch(\nu_C(c) \cup \{s\}, R(c))R(c)\nu_C(c)R(c)Ch(\nu_C(c) \cup \{s\}, R(c))$.

Therefore, ν is prematching. Now let us show that ν respects quotas, i.e. that if $\nu \in \mathcal{V}$, then $T\nu \in \mathcal{V}$. Recall that $\nu_C(c) = Ch(U(c,\nu), R(c))$, and by construction of $R(c)$ the smallest set that is preferable to $\emptyset$ is the set that contains at least $\underline{q}(a)$ elements. Then the fact that if $\nu_C(c) \neq \emptyset$ contains at least $\underline{q}(a)$ elements follows from individual rationality of $\nu_C(c)$, i.e. $\nu_C(c) = Ch(\nu_C(c), R(c))$. Then $\nu_C(c)R\emptyset$, i.e. ν respects quotas and is a matching. □

Lemma 2. $\mathcal{E} \subseteq \mathcal{C}$.

Proof. From Lemma 1 we know that $\nu = T\nu$ is individually rational matching. To prove that ν is in the core we assume that there is matching ν' that dominates ν and construct a contradiction. By assumption, there is $S' \neq \emptyset$ and $C' = \bigcup_{s \in S'} \nu_S(s) \subseteq C$, such that $\nu'(a)R(a)\nu(a)$ for every $a \in C' \cup S'$, and $\nu'(a)P(a)\nu(a)$ for some $a \in C' \cup S'$.

Then, without loss of generality assume there is $\bar{c} \in C'$, such that $\nu'(\bar{c})P(\bar{c})\nu(\bar{c})$. By individual rationality we know that $Ch(\nu(\bar{c}) \cup \nu'(\bar{c}), P(\bar{c})) \nsubseteq \nu(\bar{c})$. Let $\bar{s} \in \nu'(\bar{c}) \setminus \nu(\bar{c})$, then $\bar{s} \in Ch(\nu(\bar{c})\cup\nu'(\bar{c})\cup\{\bar{s}\}, P(\bar{c})) = Ch(\nu(\bar{c})\cup(\nu'(\bar{c})\cup\{\bar{s}\}), P(\bar{c}))$. If we denote $S'' = \nu'(\bar{c})\cup\{\bar{s}\}$ then $\bar{c} \in V(\bar{s},\nu)$ by definition of $V(\bar{s},\nu)$.

Since $\bar{s} \in \nu'(\bar{c})\setminus\nu(\bar{c})$, $\bar{s} \in S'$ and $\nu'(\bar{s}) \neq \nu(\bar{s})$. Hence, by antisymmetry of $R(\bar{s})$, $\bar{c}P(\bar{s})\nu(\bar{s})$. At the same time $\nu(\bar{s}) \cup \bar{c} \subseteq V(\bar{s},\nu)$. Since ν is a fixed point, $\nu(\bar{s}) = Ch(V(\bar{s},\nu), R(\bar{c}))R\bar{c}$. Hence, $\nu(\bar{s})P(\bar{s})\nu(\bar{s})$ that is a contradiction. □

We have now shown that every fixed point of T is an element of the core. Now we need to move on to the second part and show that T is a monotone operator over the lattice of assignments that satisfy quotas. To define the lattice over $\mathcal{V}$ we need to define a partial order over $\mathcal{V}$ with respect to which T is monotone.

Definition 7. *Define*

- $\leq_C$ *on* $\mathcal{V}_C$ *by* $\nu'_C \leq_C \nu_C$ *if for every* $c \in C$ $\nu_C(c)R(c)\nu'_C(c)$.
  The strict part of $\leq_C$ *is* $<_C$ *is* $\nu'_C <_C \nu_C$ *if* $\nu'_C \leq_C \nu_C$ *and* $\nu_C \neq \nu'_C$.
- $\leq_S$ *on* $\mathcal{V}_S$ *by* $\nu'_S \leq_S \nu_S$ *if for every* $s \in S$ $\nu_S(s)R(s_i)\nu'_S(s)$.
  The strict part of $\leq_S$ *is* $<_S$ *is* $\nu'_S <_S \nu_S$ *if* $\nu'_S \leq_S \nu_S$ *and* $\nu_S \neq \nu'_S$.
- $\leq_{CS}$ *on* $\mathcal{V}$ *by* $\nu' \leq_{CS} \nu$ *if* $\nu'_C \leq_C \nu_C$ *and* $\nu_S \leq_S \nu'_S$.
  The strict part of $\leq_{CS}$ *is* $<_{CS}$ *is* $\nu' <_C \nu$ *if* $\nu' \leq_{CS} \nu$ *and* $\nu \neq \nu'$.
- $\leq_{SC}$ *on* $\mathcal{V}$ *by* $\nu' \leq_{SC} \nu$ *if* $\nu \leq_{CS} \nu'$.

Note that $\leq_{CS}$ introduces the opposition of interests between courses and students, since $\nu \leq_{CS} \nu'$ requires ν' to be more preferred by all courses and less preferred by all students. And $\leq_{SC}$ is the reverse of the partial order $\leq_{CS}$.

Lemma 3. *Let* $\mathcal{R}$ *be set strongly substitutable and* $\nu, \nu' \in \mathcal{V}$. *Then,* $\nu \leq_{CS} \nu'$ *implies that for any* $s \in S$ *and* $c \in C$*:* $U(c,\nu) \subseteq U(c,\nu')$ *and* $V(s,\nu') \subseteq V(s,\nu)$.

Proof. ($V(s,\nu') \subseteq V(s,\nu)$). If $V(s,\nu') = \emptyset$, then the claim is trivially correct. Therefore, assume that $V(s,\nu) \neq \emptyset$, then there is $c \in V(s,\nu')$. By the definition of V: $s \in Ch(\nu'(c) \cup S', R(c))$, note that since $V(s,\nu')$ is non-empty then it has at least $\underline{q}(c)$ elements. Since $\nu \leq_{CS} \nu'$, then $\nu'(c)R(c)\nu(c)$, then by set strong substitutability $s \in Ch(\nu(c) \cup S', R(c))$. In this case to guarantee that $|\nu(c) \cup S'| \geq \underline{q}(a)$ one can simply define $S'' = \nu'(c) \cup S'$ that on its own has more than $\underline{q}(a)$ elements and complete the same reasoning with S'' instead of S'. Hence $c \in V(s,\nu)$.

($U(c,\nu) \subseteq U(c,\nu')$). If $s \in U(c,\nu)$, then $cR(s)\nu(s)$. But $\nu(s)R(s)\nu'(s)$, hence $cR(s)\nu'(s)$. Therefore, $s \in U(c,\nu')$. □

Let $\mathcal{V}' = \{\nu : \nu(a)R(a)\emptyset, \ \forall a \in C \cup S\}$, note that for any $\nu \in \mathcal{V}$, $T\nu \in \mathcal{V}'$.

Lemma 4. *If* $\mathcal{R}$ *is set strongly substitutable, then restricted operator* $T|_{\mathcal{V}'}$ *is a monotone map over* $\mathcal{V}'$ *endowed with* $\leq_{CS}$ *(*$\leq_{SC}$*).*

Proof of Lemma 4 is similar to the proof of Lemma 14.3 from Echenique and Oviedo (2006), therefore is moved to the Appendix. Now we can easily prove Theorem 1 using the Lemmata.

Proof of Theorem 1. Note that $T(\mathcal{V}) \subseteq \mathcal{V}'$, then restricted operator $T|_{\mathcal{V}'} : \mathcal{V}' \to \mathcal{V}'$ is a monotone operator (by Lemma 4) and $\mathcal{V}'$ is a complete lattice[10] and $\mathcal{E} \subseteq \mathcal{V}'$. Then by Tarski's fixed point theorem $(\mathcal{E}, \leq_{CS})$ is a non-empty complete lattice. From Lemma 1 we know that $\mathcal{E} \subseteq \mathcal{C}$, therefore core is non-empty. □

3.2 Core as Set of Fixed Points

Let us characterize the core of many-to-one matching problem with quotas as set of fixed points of T.

Corollary 1. *$\mathcal{C}$ is non-empty if and only if $\mathcal{E}$ is non-empty.*

Note that the core matching is not necessary maximal, i.e. there may be students and courses that remain unmatched. Moreover, the matching in the core can be trivial - that is all agents are unmatched.

Lemma 2 shows that set of all fixed points is a subset of the core. So to prove the Corollary 1 we need only to show that set of all fixed points is a super-set of the core.

Proof. We need to prove that $\mathcal{C} \subseteq \mathcal{E}$. Together with Lemma 2 it would imply that $\mathcal{C} = \mathcal{E}$. To prove that $\mathcal{C} \subseteq \mathcal{E}$ assume to the contrary that is $\mu \in \mathcal{C}$ and $\mu \notin \mathcal{E}$. Fix $c \in C$, such that $\mu(c) \neq Ch(U(c,\mu), R(c))$, then let $\mu'(c) = Ch(U(c,\mu), R(c))$. Hence, $\mu'(c)P(c)\mu(c)$, because $\mu(c) \subseteq U(c,\mu)$. Now let $\mu'(s) = c$ for every $s \in \mu'(c)$. Let $\forall \bar{s} \in S \setminus \mu'(c)\ \mu'(\bar{s}) = \emptyset$ and $\forall \bar{c} \in C \setminus \{c\}\ \mu'(c) = \emptyset$.

Now let us show that μ' is an individually rational matching that dominates μ. Note, that it is a matching by construction, $\mu'(c)$ satisfies quotas, since it is obtained using the truncated preference relation, and for the rest of courses $\mu'(\bar{c}) = \emptyset$. Matching μ' is individually rational, since for every agent $a \in (C \cup S) \setminus (\mu(c) \cup c)\ \mu'(c) = \emptyset$ and it will be chosen from the set of alternatives that contains $\emptyset$ only. For c it is individually rational by construction (as a chosen set from $U(c,\mu)$ and for every $s \in \mu'(c)$ it is individually rational since $cR(s)\mu(s)R(s)\emptyset$, since μ is individually rational. To show that μ' dominates μ denote by $C' = \{c\}$ and by $S' = \mu'(c)$. Then for every $s \in S'\ \mu'(s)R(s)\mu(s)$ and for $c\ \mu'(c)P(c)\mu(c)$.

Therefore, $\mu \notin \mathcal{C}$ that contradicts the assumption we made in the beginning. Hence, we have shown that $\mathcal{C} \subseteq \mathcal{E}$. It implies that $\mathcal{C} = \mathcal{E}$ and concludes the proof. □

Recall that proof of Theorem 1 required set strong substitutability of preferences since we were using Tarski's fixed point theorem. Corollary 1 does not require set strong substitutability to completely characterize the core as a set of fixed points of T. This implies that all stable matchings can be found using the T-algorithm, just using different initial conditions. Note that if T-algorithm stops it stops at a fixed point of the T operator. However,

[10] Since it is a finite lattice, and every finite lattice is complete.

the set of fixed points may be empty the core is empty as well, and the T-algorithm would cycle starting from any initial condition.

3.3 Lattice Structure of the Core

Further we can specify the structure of the core, using the Theorem 1. However, to define lattice it is necessary to establish a partial order. Further we will be using order from Definition 7 but only over the set of matchings ($\mathcal{M}$) or the core ($\mathcal{C} \subseteq \mathcal{M}$).

Corollary 2. *If $\mathcal{R}$ is set strongly substitutable, then $(\mathcal{C}, \leq_{CS})$ and $(\mathcal{C}, \leq_{SC})$ are non-empty lattices.*

Note that in proof of Theorem 1 we have already shown that $\mathcal{E}$ is a non-empty lattice over $\mathcal{M}$ endowed with order $\leq_{CS}$ (or $\leq_{SC}$). And in the proof of Corollary 1 we shown that $\mathcal{C} = \mathcal{E}$.

Note that in the absence of set strong substitutability of $\mathcal{R}$ we can not guarantee the lattice structure of the core. Even if core is non-empty (T has at least one fixed point), we still can not guarantee that $\mathcal{E}$ is a lattice, since we can not apply Tarki's fixed point theorem in the absence of set strong substitutability.

4 Discussion

Echenique and Oviedo (2004) also uses the fixed point approach to characterize the core of many-to-one matching problem without quotas. They introduce slightly different operator T_{EO} and show that the set of fixed points of T_{EO} is equal to the core.[11] If the "correct" preference profile is inputted into T_{EO}, then it characterizes core of the problem with quotas as well as T. However, the sufficient condition for the non-emptiness of the core proposed by Echenique and Oviedo (2004) can not hold for the problem with quotas. Hence, T operator is needed to prove the sufficient condition for non-empticess of the core for the problem with quotas. Echenique and Oviedo (2004) show that if preference profile is *substitutable*, then the core of many-to-one matching problem without quotas is non-empty. So, let us show that this condition can not be applied for the many-to-one problem with quotas.

Definition 8. *$R(c)$ for given $c \in C$ satisfies* ***substitutability*** *if for any $S' \subset S$, containing $s, \bar{s} \in S'$, $s \in Ch(S', R(c))$ implies $s \in Ch(S' \setminus \{\bar{s}\}, R(c))$.*

A preference profile $\mathcal{R}$ is **substitutable** if $R(c)$ satisfies substitutability for every course $c \in C$.

[11] This result is equivalent to the Lemma 1

Lemma 5. *If there is a course $c \in C$, such that $\underline{q}(c) \geq 2$, then $\mathcal{R}$ is not substitutable.*

Lemma 5 shows that if there is at least one course quota at least two, then the course's truncated preference relation violates substitutability. Hence, the condition from Echenique and Oviedo (2004) can not be applied for the problem with quotas.

Example: Let $C = \{c_1, c_2\}$, $S = \{s_1, s_2\}$, $\underline{q}(c_1) = \underline{q}(c_2) = \bar{q}(c_1) = \bar{q}(c_2) = 2$.

$$\begin{aligned}
R(s_1) &: \; c_1 P(s_1) c_2 \\
R(s_2) &: \; c_2 P(s_2) c_1 \\
\tilde{R}(c_1) &: \; \{s_1, s_2\} P(c_1) s_1 P(c_1) s_2 \\
\tilde{R}(c_2) &: \; \{s_1, s_2\} P(c_2) s_2 P(c_2) s_1 \\
R(c_1) &: \; \{s_1, s_2\} \\
R(c_2) &: \; \{s_1, s_2\}
\end{aligned}$$

We can illustrate this using $R(c_1) : \{s_1, s_2\}$, the preference relation trivially satisfies set strong substitutability. And at the same time it violates substitutability, since $\{s_1, s_2\} = Ch(\{s_1, s_2\}, R(c_1))$, then by substitutability s_1 should be equal to $Ch(\{s_1\}, R(c_1))$, while we know that $\emptyset = Ch(\{s_1\}, R(c_1))$. At the same time $\mathcal{R}$ *satisfies set strong substitutability* and we can guarantee non-emptiness of the core using Theorem 1.

As we mentioned above T_{EO} characterizes core of the problem with quotas if a "correct" preference profile is inputted. Let us show what do we mean by the "correct" preference profiles and what happens if the "wrong" preference profile is used. If T_{EO} uses $\tilde{R}(c)$ instead of truncated $R(c)$, hence it would return the core of the many-to-one matching problem without quotas. The problem without quotas[12] has the unique core element μ^{NQ}, that is $\mu^{NQ}(c_1) = s_1$ and $\mu^{NQ}(c_2) = s_2$.[13] While the problem with quotas has two elements in the core: μ^{Q1}: $\mu^{Q1}(c_1) = \{s_1, s_2\}$, $\mu^{Q2}(c_2) = \emptyset$; and μ^{Q2}: $\mu^{Q2}(c_1) = \emptyset$, $\mu^{Q2}(c_2) = \{s_1, s_2\}$. Therefore, T_{EO} would return μ^{NQ} since it is an element of the core of matching problem without quotas. But it is not a matching in the case of problem with quotas.

If T_{EO} would take as input the truncated preference relation of courses ($R(c)$) instead of $\tilde{R}(c)$, then it would return the core elements for the problem with quotas. For this purpose let us explicitly define the T_{EO} operator. Denote by $V_{OE}(s, \nu) = \{c \in C : s \in Ch(\nu(c) \cup \{s\}, R(c))\}$ for any $s \in S$, then

$$(T_{EO}\nu)(a) = \begin{cases} Ch(U(x,\nu), R(a)) \; if \; a \in C \\ Ch(V_{OE}(x,\nu), R(a)) \; if \; a \in S \end{cases}$$

[12]Problem without quotas uses $\tilde{R}(c)$ as preferences of courses.

[13]Since it is matching we show only one side of the assignment, and the second part is induced: $\mu^{NQ}(s_1) = c_1$ and $\mu^{NQ}(s_2) = c_2$.

According to the preference profile $R(c)$ the matching μ^{NQ} becomes unacceptable, therefore can not be element of the core. And it can be easily shown that if we consider problem without quotas, but with $R(c)$ instead of $\tilde{R}(c)$, then the core would contain of two elements: μ^{Q1} and μ^{Q2}. Therefore, the T_{EO} operator has two fixed points which are elements of the core. Moreover, taking $T_{EO}\nu$ is faster than taking $T\nu$, because T requires considering all subsets of S to construct $V(c,\nu)$, while to construct $V_{OE}(c,\nu)$ it is enough just to consider all elements of S. Hence, taking $T_{EO}\nu$ can be done in polynomial time, while $T\nu$ can not be done in polynomial time.

Let us conclude with a remark on applicability of T and T_{EO} operators for unrestricted preference profiles and determination of non-emptiness of the core in many-to-one matching problem with quotas. Biró et al. (2010) shows that the problem of determining whether the many-to-one matching problem with quotas has a non-empty core or not is NP-complete. And neither T nor T_{EO} can be used to solve the problem in the polynomial time, since they require starting algorithm from any possible initial point (an element of $\mathcal{V}$) to determine whether there are any elements in the core.

Note that even if we know that core is non-empty searching for an element of it may be not a feasible problem.[14] Even assuming the set strong substitutability of preferences would not resolve the problem, since in the example above both T and T_{EO} would cycle starting from any assignment that is not an element of the core. Hence, T and T_{EO} algorithms may not be able to solve the large scale problem in feasible time. However, the matching mechanisms are widely applied for large scale markets.

It may be that there is some compromise between how restrictive a solution is and how fast it can be reached. Perhaps finding a core that is easily calculated would require uselessly implausible restrictions on preferences. If there is such a trade-off, then many algorithms could be useful to fill out the frontier of possibilities. A larger menu would be more likely to provide the right mix of assumptions and computational requirements for each application

Appendix: Proofs

Proof of Lemma 4. We need to prove that whenever $\nu \leq_{CS} \nu'$, then $T\nu \leq_{CS} T\nu'$. Let $\nu \leq_{CS} \nu'$, fix $c \in C$ and $s \in S$. The proof consists of two parts, we need to show that $(T\nu')(c)R(c)(T\nu)(c)$ and $(T\nu)(s)R(s)(T\nu')(s)$.

($(T\nu')(c)R(c)(T\nu)(c)$**).**

[14] Since preference profile can not be substitutable, the complexity result from Echenique and Oviedo (2004) is not applicable.

From Lemma 3 we know that $U(c,\nu) \subseteq U(c,\nu')$. Then we can show that:

$$Ch(U(c,\nu'),R(c)) = Ch([Ch(U(c,\nu'),R(c)) \cup Ch(U(c,\nu),R(c))],R(c))$$

Let $X \subseteq Ch(U(c,\nu'),R(c)) \cup Ch(U(c,\nu),R(c))$, then $X \subseteq U(c,\nu') \cup U(c,\nu) = U(c,\nu')$. Therefore, $Ch(U(c,\nu'),R(c))R(c)X$ and at the same time $Ch(U(c,\nu'),R(c)) \subseteq Ch([Ch(U(c,\nu'),R(c)) \cup Ch(U(c,\nu),R(c))],R(c))$. Therefore we shown that $Ch(U(c,\nu'),R(c)) = Ch([Ch(U(c,\nu'),R(c)) \cup Ch(U(c,\nu),R(c))],R(c))$. Note that $(T\nu')(c) = Ch(U(c,\nu'),R(c))$ and $(T\nu)(c) = Ch(U(c,\nu),R(c))$, then:

$$(T\nu')(c) = Ch([(T\nu')(c) \cup (T\nu)(c)],R(c))$$

Hence, $(T\nu')(c)R(c)(T\nu)(c)$.

($(T\nu)(s)R(s)(T\nu')(s)$).
From Lemma 3 we know that $V(c,\nu') \subseteq V(c,\nu)$. Then, $(T\nu)(s) = Ch(V(c,\nu),R(s))\ R(s)\ Ch(V(c,\nu'),R(s)) = T(nu')(s)$.

Therefore, $T\nu \leq_{CS} T\nu'$, by the definition of $\leq_{CS}$. Similar proof can be conducted for $\leq_{SC}$. □

Proof of Lemma 5. If there is $c \in C$ such that $\tilde{R}(c)$ does not satisfy substitutability, then $R(c)$ does not satisfy substitutability. Therefore, assume that for every $c \in C$ $\tilde{R}(c)$ satisfies substitutability. The fix $c \in C$ and $s \in Ch(S^0, R(c)$ for some $S^0 \subseteq S$ such that $\underline{q}(c) \leq |S^0| \leq \bar{q}(c)$. Take $\bar{s}^0 \in S^0 \setminus \{s\}$, and consider $S^1 = S^0 \setminus \{\bar{s}^0\}$, then by substitutability of $R(c)$ $s \in Ch(S^1, R(c))$. We can repeat this procedure until $2 \leq |S^k| < \underline{q}(c)$, then $s \notin Ch(S^k \setminus \{\bar{s}^k\}, R(c))$, that violates substitutability. □

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