Testable Implications of Quasi-Hyperbolic and Exponential Time Discounting [1]

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May 19, 2014

[1]We thank Kim Border and Chris Chambers for inspiration and advice. This paper subsumes the paper Echenique and Saito (2013b) "Testable Implication of Exponential Discounting," circulated as Caltech Social Science Working Paper 1381.

Abstract

We present the first revealed-preference characterizations of the models of exponential time discounting, quasi-hyperbolic time discounting, and other time-separable models of consumers' intertemporal decisions. The characterizations provide non-parametric revealed-preference tests, which we take to data using the results of a recent experiment conducted by Andreoni and Sprenger (2012). For such data, we find that less than half the subjects are consistent with exponential discounting, and only a few more are consistent with quasi-hyperbolic discounting.

1 Introduction

This paper presents an investigation into the observable consequences of the standard model of exponential discounting, the model of quasi-hyperbolic discounting, and its generalizations.

Consider an agent who chooses among dated consumptions of a single good, a good that one can think of as money. Such agents populate many models in various areas of economics; for example macroeconomics and finance. One theory is that the agent has a utility function $U(x_0, \ldots, x_T)$ for the consumption of x_t in each date t. If one is given data on the choices of our agent, the Strong Axiom of Revealed Preference (SARP) tells us whether the choices are consistent with some utility function U that could explain the data.

Suppose instead that we have a more specific theory in mind. Say that we conjecture that U takes the form of exponential discounting:

$$U(x_0, \ldots, x_T) = \sum_{t=0}^{T} \delta^t u(x_t).$$

We term this model EDU for short. *What is the version of SARP that allows us to decide whether data is consistent with such a utility function?* The version of the revealed preference axiom that characterizes EDU is obviously going to be stronger than SARP, but the literature on revealed preferences does not (until now) provide an answer.

We can instead imagine a more general model than exponential discounting, the model of quasi-hyperbolic discounting (QHD; Phelps and Pollak, 1968; Laibson, 1997):

$$U(x_0, \ldots, x_T) = u(x_0) + \beta \sum_{t=1}^{T} \delta^t u(x_t).$$

What is the version of SARP that allows us to decide whether data is consistent with quasi-hyperbolic discounting?

The theoretical contribution of this paper is to provide answers to these questions. We provide the first revealed-preference characterizations of the models of exponential and quasi-hyperbolic discounting: defining revealed preference axioms (axioms like SARP, but stronger) that are satisfied by data if and only if the data is consistent with each of the aforementioned models.

Our main result is that a certain revealed preference axiom, termed the "Strong Axiom of Revealed Exponentially Discounted Utility" (SA-EDU), describes the choice data that are consistent with *convex EDU preferences.* SA-EDU builds on the simplest implication of consumption smoothing on the relation between prices and quantities: that demand slopes down. The axiom constraints quantities and prices in a way that generalizes downward-sloping demand, but accounting for the different unobservable components in EDU.

SA-EDU seems like a relatively weak imposition on data, in the sense that it constraints prices and quantities *in those situations in which unobservables do no matter.* Essentially, SA-EDU requires one to consider situations in which unobservables "cancel out", and check that the implications of concave utility on prices are not violated.

Aside from EDU, the paper also includes a revealed preference characterization of QHD; and of more general models, namely time-separable utility, time-dependent discounting, and minor variations on these models. All of our characterizations are based on ideas similar to SAR-EDU, building on the implications of consumption smoothing for each of these models.

Building on the theory developed in the paper, we seek to make an empirical contribution by applying our revealed preference axioms to data from an insightful recent experiment conducted by Andreoni and Sprenger (2012) (hereafter AS). AS's experimental design fits our setup very well, and we can apply our tests to their data. The contribution is not only to show that our axioms are readily applicable and useful, but we also hope to contribute to the substantive debate on the role of exponential and quasi-hyperbolic discounting.

The following is a summary of our findings using the data from Andreoni and Sprenger's experiment.

Individual-level pass rates. We check for the consistency of individual subject's choices with the various revealed preference axioms. We find that 47% of the subjects are rationalizable using exponential discounting (EDU). Turning to quasi-hyperbolic discounting (QHD), 49% of the subjects are rationalizable using a quasi-hyperbolic functional form for their utility.

Note that all EDU rationalizable subjects are also QHD rationalizable, so the scope of QHD is therefore not much more than EDU on this dataset. Only two subjects are rationalizable as QHD, but not rationalizable as EDU. In fact, when we look at subjects who are not rationalizable, their behavior is in some sense as close to EDU as it is to QHD (see Section 6.2 for the precise meaning of this). So there is some evidence that QHD does not explain much more than EDU.

Our methodology allows us to go beyond QHD, and establish that 67% of subjects are consistent with a utility function that is time-separable: the remaining subjects are inconsistent with time-separability, although their choices are rationalizable using some utility function (they satisfy SARP, but this is by design of Andreoni and Sprenger's experiment). Section 6 below has more details.

Similarities and differences with AS's findings. The results from applying our non-parametric tests correlate well with some of AS's basic findings, but give a somewhat different interpretation of others. Very roughly speaking, while AS find moderate support for EDU, our conclusion is closer to a rejection of EDU.

AS estimate a utility with quasi-hyperbolic discounting: $u(x_0) + \beta \sum_{t=1}^{T} \delta^t u(x_t)$. For almost all the subjects who pass our test, the AS estimates for β are roughly equal to 1, indicating exponential (non-hyperbolic) discounting. This means that the two methodologies have a lot in common, as they detect the same subjects as exponential discounters.

The counterpart of this finding is that the majority of subjects who fail our EDU test also have estimates of β that differ from 1. This result, however, should be taken with a grain of salt because most of the subjects who fail the EDU test also fail our test for quasi-hyperbolic discounting. Therefore a utility function with quasi-hyperbolic discounting could be considered mispecified.

It is also interesting to note that the estimated values of β (provided by AS) for subjects who failed our EDU test are symmetrically distributed around 1. The "average" subject therefore looks, in some sense, as an exponential discounter; even though most agents are not really consistent with that model. It is possible that AS's aggregate estimates, which are supportive of EDU, reflect the behavior of such an average subject.

The role of corner solutions. Among subjects who satisfy our EDU axiom, and are therefore rationalizable with a utility function with exponential discounting, there is a disproportionate number of corner solutions. Many of the subjects who are consistent with the EDU model also make choices at the corners of the budget set. Specifically, they make choices where they spend all their budget on a reward that is obtained later in time.

This raises a concern, because our test for EDU rationality seems to have lower power at corner solutions. We ran a simulation of agents that are severely present- or future-biased quasi-hyperbolic discounters, but that have *linear utility.* So their choices tend to be at the corners of the budget set. Basically all the observations in our simulations are consistent with the EDU revealed preference axiom, and hence would be deemed to be EDU rational.

Of course, a strict application of the philosophy behind revealed preferences would say that our simulation results do not matter: the agents in our simulation behave as if they are EDU rational and that is all that is really postulated by EDU as a positive theory of economic behavior.

This interpretation is a matter of debate, however. We feel that the lack of power is an interesting, and potentially important, phenomenon when testing for EDU. Some researchers may conclude that the pass rate of 47% for EDU is really an overestimate, and that in reality fewer subjects are likely to be EDU rational.

Related literature. There are different behavioral axiomatizations of EDU in the literature, starting with Koopmans (1960), and followed by Fishburn and Rubinstein (1982), Fishburn and Edwards (1997), and Bleichrodt et al. (2008). All of them take preferences as primitive, or in some cases they take utility over consumption streams as the primitive. The idea is that an analyst can observe all pairwise comparisons of consumption streams, or that the relevant behavior consists of all pairwise comparisons of consumption streams. Note that this assumes knowledge of an infinite number of pairwise comparisons: so the given "dataset" is infinite.

Koopmans (1960) proposes the well-known stationarity axiom, which says a preference is not affected if a common first consumption is dropped and the timing of all other consumptions is advanced by one period. The stationarity axiom is used by many other authors, and the axiom is used together with the assumption that the set of periods is infinite; indeed it requires infinite time. Our axiomatization is the first in an environment where agents choose finite consumption streams.

In Fishburn and Rubinstein (1982) preferences are defined on one-time consumptions in continuous time. In Fishburn and Edwards (1997), preferences are defined on infinite consumption streams that differ in at most finitely many periods. More recently, Bleichrodt et al. (2008) show that Koopmans (1960)'s axioms imply the boundedness of the utility function. Then, Bleichrodt et al. (2008) axiomatize the EDU model possibly with unbounded utility function by using preferences defined on infinite consumption streams.

The quasi-hyperbolic discounting model was first proposed by Phelps and Pollak (1968), who did not characterize its behavioral consequences. There are several more recent studies that present a behavioral characterization of QHD, but all take preferences and infinite time horizons as their primitives, and therefore differ from our results. See Attema et al. (2010) and Montiel Olea and Strzalecki (forthcoming).

In terms of expenditure data, Browning (1989) gives a revealed-preference axiom for the EDU model with no discounting ($\delta = 1$). Crawford (2010) investigates intertemporal consumption and discusses a particular violation from non-separability (TSU), namely habit formation. Crawford presents Afriat inequalities for the model of habit formation, and uses Spanish consumption data to carry out the test (see also Crawford and Polisson, 2014). Blow et al. (2013) present a test for quasi-hyperbolic discounting based on solving Afriat inequalities. They also apply their model to Spanish consumption data. It is important to emphasize that these models, starting with Browning (1989) allow for the existence of many goods in each period; but they do not allow for more than one (intertemporal) purchase for each agent. The reason is that these authors consider consumption data with one purchase for each agents (or household) and each period. We have instead assumed that there is only one good (money) in each period; but we allow for more than one purchase per agent. This is crucial in order to apply our tests to the experimental data obtained by Andreoni and Sprenger (2012).

On the empirical side, several approaches have been proposed to identify, estimate, or calibrate time preferences (see Frederick et al. (2002) for a comprehensive overview of this literature until the early 21st century). In one strand of the literature, researchers estimate discount rates using filed data on consumption and credit card borrowings in the context of a life-cycle consumption model (Cagetti, 2003; Carroll and Samwick, 1997; Gourinchas and Parker, 2002; Laibson et al., 2003, 2007) or durable goods purchasing (Gately, 1980; Hausman, 1979). Another strand of the literature involves identification and estimation of time preferences using choice data from controlled laboratory experiments. In a typical experimental paradigm, called the Multiple Price List (MPL) method, participants are asked to make a series of binary choices between a smaller immediate payment and a larger delayed payment (Chabris et al., 2008; Coller and Williams, 1999; Harrison et al., 2002; Kirby et al., 1999). This literature is often subject to the critique that discount rates elicited using MPL method deviate from those estimated using field data, possibly due to, among other things, an implicit assumption of linear utility in experimental studies (Frederick et al., 2002). In order to correct this bias, Andersen et al. (2008) propose the Double Multiple Price List (DMPL) method, in which two MPLs, each of which is designed to measure time preferences and risk preferences, are administered. As an alternative solution, Andreoni and Sprenger (2012) propose the Convex Time Budget (CTB) method, in which participants are asked to choose from a convex, intertemporal budget set. [1] They find a reasonable level of discount rate and utility curvature, and evidence in favor of dynamically-consistent time preferences. [2] [3]

[1] Several recent experimental studies use the CTB design both in the laboratory and in the field setting, including Andreoni et al. (2013b), Augenblick et al. (2013), Barcellos and Carvalho (2014), Carvalho et al. (2013), Carvalho et al. (2014), and Giné et al. (2013).

[2] Aside from these methodological advances, recent experimental studies call for attention on credit constraints and background consumptions in measuring time preferences (Ambrus et al., 2014; Dean and Sautmann, 2014). An insightful paper by Halevy (2014) nonparametrically identifies time-consistent, stationary, and time-invariant choices in recurrent classroom experiments and finds that not only present bias but time-varying preferences are important driver of time inconsistent choices.

[3] Relationships between experimentally elicited time preferences (especially present-biasedness) and measures of real-life economic behavior (e.g., credit card borrowing) and demographic variables (e.g., age, smok-

2 Models

For $T > 0$, we abuse notation and use T to denote the set $\{0, 1, \ldots, T\}$. In our model, T will be the (finite) duration of time, and consumption streams will be sequences $(x_0, \ldots, x_T) = (x_t)_{t \in T} \in \mathbf{R}^T_+$. Note that the cardinality of the set T is $T + 1$, which we do not believe leads to any confusion. The notation $\mathbf{R}^T$ means the $T + 1$ dimensional Euclidean space.

The object of choice in our model is a sequence $(x_t)_{t \in T} \in \mathbf{R}^T_+$. We assume that an agent has an income I and faces prices $p \in \mathbf{R}^T_{++}$; we can think of prices as interest rates.

The theories we investigate postulate that an agent solves the problem

$$\begin{aligned} \max \quad & U(x_0, \ldots, x_T) \\ \text{s.t.} \quad & \sum_{t \in T} p_t x_t \leq I \end{aligned}$$

We consider the following classes of utility functions. Let $\mathcal{C}$ be the set of all continuous, concave, and strictly increasing functions $u : \mathbf{R}_+ \to \mathbf{R}$.

1. *Time-separable utility* : The class TSU of all U that can be written as

$$U(x_0, \ldots, x_T) = \sum_{t \in T} u_t(x_t),$$

   for some $u_t \in \mathcal{C}$ for all $t \in T$.

2. *General time discounting* : The class GTD of all U that can be written as

$$U(x_0, \ldots, x_T) = \sum_{t \in T} D(t) u(x_t),$$

   for some $u \in \mathcal{C}$, and a function $D : T \to \mathbf{R}_+$.

3. *Monotone time discounting* : The class MTD of all U that can be written as

$$U(x_0, \ldots, x_T) = \sum_{t \in T} D(t) u(x_t),$$

   for some $u \in \mathcal{C}$, and a function $D : T \to \mathbf{R}_+$ that is monotonically decreasing.

4. *Quasi-hyperbolic discounting* : The class QHD of all U that can be written as

$$U(x_0, \ldots, x_T) = \sum_{t \in T} D(t) u(x_t),$$

ing, attitudes to risk) are also of interests (Burks et al., 2012; Dean and Ortoleva, 2014; Meier and Sprenger, 2010, 2013; Tanaka et al., 2010).

for some $u \in \mathcal{C}$, and where

$$D(t) = \begin{cases} 1 & \text{if } t = 0 \\ \beta\delta^t & \text{if } t > 0 \end{cases}$$

for some β and $\delta \in (0, 1]$.

In particular, if $\beta \leq 1$, then U is said to belong to P-QHD of *present biased* hyperbolic discounting; and if $\beta \geq 1$, then U is said to belong to F-QHD of *future biased* hyperbolic discounting.

5. *Exponential discounting* : The class EDU of all U that can be written as

$$U(x_0, \ldots, x_T) = \sum_{t \in T} \delta^t u(x_t),$$

for some $u \in \mathcal{C}$, and some $\delta \in (0, 1]$.

3 Rationalization

Definition 1. *A* dataset *is a collection* $(x^k, p^k)_{k=1}^K$*, where* $x^k \in \mathbf{R}^T_+$ *and* $p^k \in \mathbf{R}^T_{++}$.

We have considered seven classes of utility functions $U(x_0, \ldots, x_T)$. They are: TSU, GTD, MTD, QHD, P-QHD, F-QHD, and EDU; listed in order of how restrictive they are.

In the following definition, the set M of utility functions can be any of the seven classes defined above.

Definition 2. *A dataset* $(x^k, p^k)_{k=1}^K$ *is* M rational *if there is a utility function* U *in the class M of utilities such that for all* k*,*

$$p^k \cdot y \leq p^k \cdot x^k \Rightarrow U(y) \leq U(x^k)$$

4 Axioms

Definition 3. *A sequence of pairs* $(x^{k_i}_{t_i}, x^{k'_i}_{t'_i})_{i=1}^n$ *in which*

1. $x^{k_i}_{t_i} > x^{k'_i}_{t'_i}$ *for all* i*;*

2. *each* k *appears as* k_i *(on the left of the pair) the same number of times it appears as* k'_i *(on the right):*

is a regular *sequence.*

Remark 1. When $K = 1$ then $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^n$ is regular iff $x_{t_i}^{k_i} > x_{t_i'}^{k_i'}$ for all i. The $K = 1$ special case is relevant for field consumption data (not for the experimental data we have used here), as one normally observes one intertemporal decision for each agent in such data.

Strong Axiom of Revealed Time Separable Utility (SA-TSU): *For any regular sequence of pairs $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^n$ in which each $t_i = t_i'$ for all i, the product of prices satisfies that*

$$\prod_{i=1}^{n} \frac{p_{t_i}^{k_i}}{p_{t_i'}^{k_i'}} \leq 1.$$

Strong Axiom of Revealed General Discounted Utility (SA-GTD): *For any regular sequence of pairs $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^n$ in which each t appears as t_i (on the left of the pair) the same number of times it appears as t_i' (on the right), the product of prices satisfies that*

$$\prod_{i=1}^{n} \frac{p_{t_i}^{k_i}}{p_{t_i'}^{k_i'}} \leq 1.$$

Strong Axiom of Revealed Monotonic Time-Varying Discounted Utility (SA-MTD): *For any regular sequence of pairs $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^n$ in which there is a permutation π of $\{1, 2, \ldots, n\}$ such that $t_i \geq t'_{\pi(i)}$, the product of prices satisfies that*

$$\prod_{i=1}^{n} \frac{p_{t_i}^{k_i}}{p_{t_i'}^{k_i'}} \leq 1.$$

Strong Axiom of Revealed Quasi-Hyperbolic Discounted Utility (SA-QHD): *For any regular sequence of pairs $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^n$ in which*

1. $\sum_{i=1}^n t_i \geq \sum_{i=1}^n t_i'$;

2. $\#\{i : t_i > 0\} = \#\{i : t_i' > 0\}$;

The product of prices satisfies that

$$\prod_{i=1}^{n} \frac{p_{t_i}^{k_i}}{p_{t_i'}^{k_i'}} \leq 1.$$

Strong Axiom of Revealed Quasi-Hyperbolic Present-Biased Utility (SA-P-QHD): *For any regular sequence of pairs $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^n$ in which*

1. $\sum_{i=1}^n t_i \geq \sum_{i=1}^n t_i'$;

2. $\#\{i : t_i > 0\} \geq \#\{i : t'_i > 0\}$*;*

The product of prices satisfies that

$$\prod_{i=1}^{n} \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}} \leq 1.$$

Strong Axiom of Revealed Quasi-Hyperbolic Future-Biased Utility (SA-F-QHD): *For any regular sequence of pairs* $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^{n}$ *in which*

1. $\sum_{i=1}^{n} t_i \geq \sum_{i=1}^{n} t'_i$*;*

2. $\#\{i : t_i > 0\} \leq \#\{i : t'_i > 0\}$*;*

The product of prices satisfies that

$$\prod_{i=1}^{n} \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}} \leq 1.$$

Strong Axiom of Revealed Exponentially Discounted Utility (SA-EDU): *For any regular sequence of pairs* $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^{n}$ *in which* $\sum_{i=1}^{n} t_i \geq \sum_{i=1}^{n} t'_i$*, the product of prices satisfies that*

$$\prod_{i=1}^{n} \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}} \leq 1.$$

Theorem 1. *Let* $(x^k, p^k)_{k=1}^{K}$ *be a dataset. For*

$$M \in \{\mathit{TSU},\ \mathit{GTD},\ \mathit{MTD},\ \mathit{QHD},\ \mathit{P\text{-}QHD},\ \mathit{F\text{-}QHD},\ \mathit{EDU}\},$$

the dataset is M-rational iff it satisfies SA-M.

5 A derivation of SA-EDU

Consider the following maximization problem.

$$\begin{array}{ll} \max_{x \in \mathbf{R}_+^T} & \sum_{t \in T} \delta^t u(x_t) \\ \text{s.t.} & \sum_{t \in T} p_t x_t \leq I, \end{array} \tag{1}$$

For ease of exposition, suppose that the function u is continuously differentiable. The first-order condition for an interior solution is

$$\delta^{t-1} u'(x_t) = \lambda p_t,$$

where λ is a Lagrange multiplier. So if a dataset $(x^k, p^k)_{k=1}^K$ is EDU rational, the discount factor δ and utility u must satisfy the above first order condition for each x_t^k and p_t^k.

Suppose that one tries to derive the implications of the first order condition for the observed quantities and prices. From the first-order conditions, one can obtain that

$$\frac{u'(x_{t'}^{k'})}{u'(x_t^k)} = \frac{\delta^t}{\delta^{t'}} \frac{\lambda^{k'} p_{t'}^{k'}}{\lambda^k p_t^k}.$$

Suppose that $x_t^k > x_{t'}^{k'}$. The concavity of u and $x_t^k > x_{t'}^{k'}$ implies that

$$\frac{\delta^t}{\delta^{t'}} \frac{\lambda^{k'} p_{t'}^{k'}}{\lambda^k p_t^k} \leq 1,$$

but the discount rate δ and the Lagrange multipliers $\lambda^{k'}$ and λ^k are unobservable so we cannot conclude anything about the observable $p_{t'}^{k'}/p_t^k$.

There is, however, one implication of EDU and the concavity of u that can unambiguously be obtained, despite the role of unobservables. We can consider a sequence of pairs $(x_t^k, x_{t'}^{k'})$ chosen such that when we divide first-order conditions as above, all Lagrange multipliers cancel out, and the effect of the discount factors is unambiguous (even though we do not know the value of the discount factor). For example, consider

$$x_{t_1}^{k_1} > x_{t_2}^{k_2} \text{ and } x_{t_3}^{k_2} > x_{t_4}^{k_1}.$$

such that

$$t_1 + t_3 \geq t_2 + t_4.$$

By manipulating first-order conditions we obtain that:

$$\frac{u'(x_{t_1}^{k_1})}{u'(x_{t_2}^{k_2})} \cdot \frac{u'(x_{t_3}^{k_2})}{u'(x_{t_4}^{k_1})} = \left(\frac{\delta^{t_2-1}}{\delta^{t_1-1}} \frac{\lambda^{k_1} p_{t_1}^{k_1}}{\lambda^{k_2} p_{t_2}^{k_2}} \right) \cdot \left(\frac{\delta^{t_4-1}}{\delta^{t_3-1}} \frac{\lambda^{k_2} p_{t_3}^{k_2}}{\lambda^{k_1} p_{t_4}^{k_1}} \right) = \delta^{(t_2+t_4)-(t_1+t_3)} \frac{p_{t_1}^{k_1}}{p_{t_2}^{k_2}} \frac{p_{t_3}^{k_2}}{p_{t_4}^{k_1}}$$

Notice that the pairs $(x_{t_1}^{k_1}, x_{t_2}^{k_2})$ and $(x_{t_3}^{k_2}, x_{t_4}^{k_1})$ have been chosen so that the Lagrange multipliers would cancel out and the discount factors unambiguously increase the value on the left hand side (i.e., $\delta^{(t_2+t_4)-(t_1+t_3)} \geq 1$ for any $\delta \in (0, 1]$).

Now the concavity of u and the assumption that $x_{t_1}^{k_1} > x_{t_2}^{k_2}$ and $x_{t_3}^{k_2} > x_{s_4}^{k_1}$ imply that the product $\delta^{(t_2+t_4)-(t_1+t_3)}(p_{t_1}^{k_1}/p_{t_2}^{k_2})(p_{t_3}^{k_2}/p_{t_4}^{k_1})$ cannot exceed 1. Since $\delta^{(t_2+t_4)-(t_1+t_3)} \geq 1$ for any $\delta \in (0, 1]$, then $(p_{t_1}^{k_1}/p_{t_2}^{k_2})(p_{t_3}^{k_2}/p_{t_4}^{k_1})$ cannot exceed 1. Thus, we obtain an implication of EDU for prices, an observable entity. *No matter what the values of the unobservable δ and u, we find that the ratio of prices cannot be more than* 1.

In general, the assumption of EDU rationality will require that, for any collection of sequences as above (appropriately chosen so that Lagrange multipliers will cancel out and the discount factors unambiguously increase the product of the ratio of prices) the product of the ratio of prices cannot exceed 1. This idea is captured by the definition of SA-EDU.

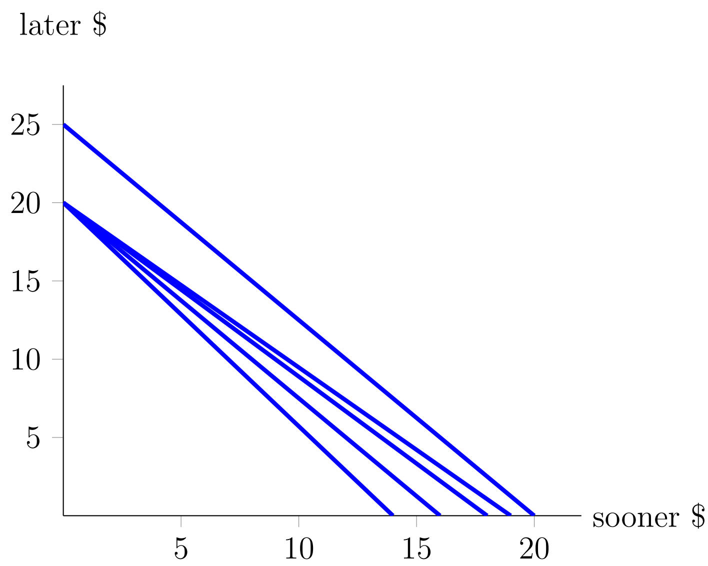


FIGURE 1: An illustration of the CTB design in Andreoni and Sprenger (2012). Budget sets are represented in blue lines, fixing one time frame at $(\tau, d) = (0, 35)$.

6 Empirical Application

6.1 Description of the Data

Andreoni and Sprenger (2012) introduce an experimental method called the Convex Time Budget (CTB) design. In contrast with the "multiple price list method" (Andersen et al., 2008), subjects in Andreoni and Sprenger (2012) are asked to allocate 100 experimental tokens between "sooner" (time τ) and "later" (time $\tau + d$) accounts.

Tokens allocated to each account have a value of a_τ and $a_{\tau+d}$, converting experimental currency unit into real monetary value for final payments. The gross interest rate over d days is thus given by $1 + r \equiv a_{\tau+d}/a_\tau$. Participants complete 45 decisions: $\tau \in \{0, 7, 35\}$ times $d \in \{35, 70, 98\}$ times 5 questions (4 values of $(1 + r)$, varying across different pairs of (τ, d); see Figure 1 for an illustration).[4]

Each participant's decision in a trial is characterized by a tuple $(\tau, d, a_\tau, a_{\tau+d}, c_\tau)$: the first four elements $(\tau, d, a_\tau, a_{\tau+d})$ characterize the budget set she faces in this trial; and c_τ is the number of tokens she decides to allocate to the sooner payment.

In the experiment, participants make a two-period utility maximization problem between

[4]Note that for a given pair of starting date and delay length (τ, d), 5 budgets are nested as illustrated in Figure 1. Looking at all 45 budget sets, except for 8 cases in which $(a_\tau, a_{\tau+d}) = (0.2, 0.25)$, $a_{\tau+d}$ is fixed at 0.2 and a_τ ranges between 0.1 and 0.2. Participants' choices therefore always satisfy GARP by design.

period τ and $\tau + d$:

$$\max \; U(x_\tau, x_{\tau+d})$$
$$\text{s.t.} \; p_\tau x_\tau + x_{\tau+d} = m.$$

In order to apply our theoretical framework, we need to imagine that participants are solving the problem

$$\max \; U(x_0, \dots x_T)$$
$$\text{s.t.} \; \sum_{t=0}^{T} p_t x_t = m.$$

We rewrite the choice data to fit the theoretical framework of Section 3. We set prices to be $p_\tau = 1 + r = a_{\tau+d}/a_\tau$ and $p_{\tau+d} = 1$ (normalization); and we define consumptions (monetary amounts) $x_\tau = c_\tau \cdot a_\tau$ and $x_{\tau+d} = (100 - c_\tau) \cdot a_{\tau+d}$. We shall implicitly set the prices of periods that are not offered to be very high, so that agents cannot afford consumption in those periods.

When participants face a convex budget with $(\tau, d) = (35, 70)$ for example, we treat prices p_t for $t \neq 30, 75$ are "extremely high" and she cannot afford any positive consumption at dates other than 30 and 30 + 75. In this way, we obtain a dataset $(x^k, p^k)_{k=1}^{45}$, $x^k \in \mathbf{R}^T_+$ and $p^k \in \mathbf{R}^T_{++}$, for each of 97 participants in the experiment.

Several features of the AS design make this dataset ideal for our exercise. First and most importantly (and obviously), the experimental setup is precisely the situation our model tries to capture: Participants choose how much to consumer from a convex budget set. As we briefly mention above, most previous experimental studies on intertemporal decision utilize an environment with discrete (in many cases, binary) choice sets. A great advantage of Andreoni and Sprenger's setup is that they specifically consider choice from convex budgets.

Secondly, every decision in this experiment is a *plan*: participants made all decisions for the present and the future at time 0, i.e., while they were in the laboratory. This allows us to examine the revealed preference axiom for QHD without distinguishing *sophisticated* and *naïve* present-biased preferences (O'Donoghue and Rabin, 1999a,b). Finally, Andreoni and Sprenger put a huge effort into equalizing the transaction costs of sooner and later payments, and minimizing the unwanted effects of uncertainty regarding future payments.

6.2 Results

We start by checking for the consistency of individual participant's choices with the various revealed preference axioms.

Recall that the classes of models we examine, EDU, QHD, MTD, GTD, and TSU, can be ordered by the tightness of the associated axioms. Essentially, we have that:

$$\text{EDU} \subset \text{P-QHD} \subset \text{MTD} \subset \text{GTD} \subset \text{TSU},$$

and that

$$\text{EDU} \subset \text{F-QHD} \subset \text{QHD} \subset \text{GTD} \subset \text{TSU},$$

as QHD is not comparable to MTD.

For this reason, when we find that a subject is EDU rational, she is of course also M rational for all other models M.

In the sequel, we shall label a participant as “M rational” if her dataset passes the revealed preference test for model M and “M non-rational” otherwise. We sometimes label a participant as “strictly M rational” for the most restrictive model M such that the agent is M rational. For example, a participant is strictly QHD rational if her dataset passes the QHD test but not the EDU test.

Figure 2 summarizes the results. We find that choice data from 47.4% of 97 participants (46 participants) are EDU rational. Quasi-hyperbolic discounting utility rationalizes 49.5% of the participants: There are only *two* participants who are strictly QHD rational, and both of them are in the class P-QHD of present-biased agents.

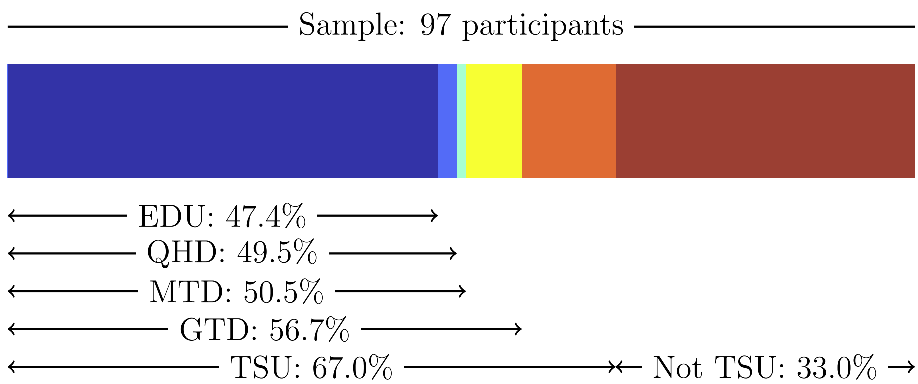


Figure 2: Subjects’ classification into models.

EDU and QHD are arguably the most important models used in economics, but it is interesting to go beyond these models and look at the more general utility functions described in Section 2.

We find that 7 additional participants (7.2%) have utility functions with time-varying discount factor (GTD), and 10 more participants (10.3%) become rational by allowing a more general, time-separable utility function (TSU). In all, 67% of subjects can be rationalized by one of the models M.

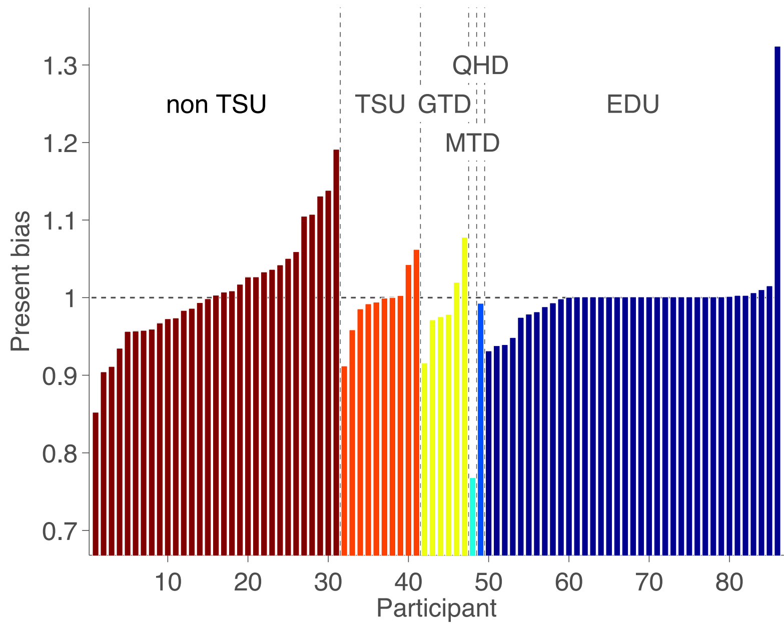


Figure 3: Estimated present-bias parameter for each category of participants.

It is striking that about 33% of participants are not rationalized by any time-separable utility functions, the most general class of model we are investigating. We cannot say, however, that those participants are irrational, since GARP is always satisfied in this dataset: Note that no budget sets in the AS experiment intersect at an interior allocation, although many budgets share the same corner in which participants receive \$20 in the delayed time periods.

Similarities and differences with AS's findings. We next revisit AS's findings and relate them to results from our nonparametric tests. AS estimate per-period discount factor, present biasedness, and utility curvature assuming a quasi-hyperbolic discounting $u(x_0) + \beta \sum_{t=1}^{T} \delta^t u(x_t)$ with a CRRA utility function $u(x) = (1/\alpha) \cdot x^{\alpha}$. [5]

In the estimation of aggregate preference parameters, AS find no evidence of present bias ($\hat{\beta} = 1.007$, $SE = 0.004$; the hypothesis of no present bias, $\beta = 1$, is not rejected, $F_{1,96} = 1.51$, $p = 0.22$). Similarly, at the individual level analysis, AS find that estimated present bias $\hat{\beta}$'s are narrowly distributed around 1 with median estimate of 1.0011.

[5]AS estimate several model specifications (e.g., assuming CARA instead of CRRA, or incorporating additional parameters to capture background consumptions), and they also use different estimation methods (e.g., two-limit Tobit model to handle corner choices). In the comparison below, we use their results from a nonlinear least squares estimation of quasi-hyperbolic discounting and CRRA utility function without background consumption.

In order to compare our non-parametric tests and the AS parametric model estimation, we use the individual level parameter estimates from AS (see Table A6-7 in Online Appendix of Andreoni and Sprenger, 2012). [6]

Our test is clearly consistent with AS's estimates: Figure 3 presents the AS-estimated present bias parameter $\hat{\beta}$ for each individual subject, *classified by the strictest test passed by the subject*. It is clear in the figure that most of the participants who pass the EDU test have estimated $\hat{\beta}$ almost equal to 1. Those who fail the EDU test but pass QHD, GTD, MTD, or TSU test tend to have $\hat{\beta} < 1$, and finally, those who do not pass any of the tests have estimated $\hat{\beta}$ which are far from 1 in magnitude compared to other groups of participants and are distributed symmetrically around 1. So our test is quite consistent with the procedure of taking an estimate of β different from one as evidence as violations of EDU.

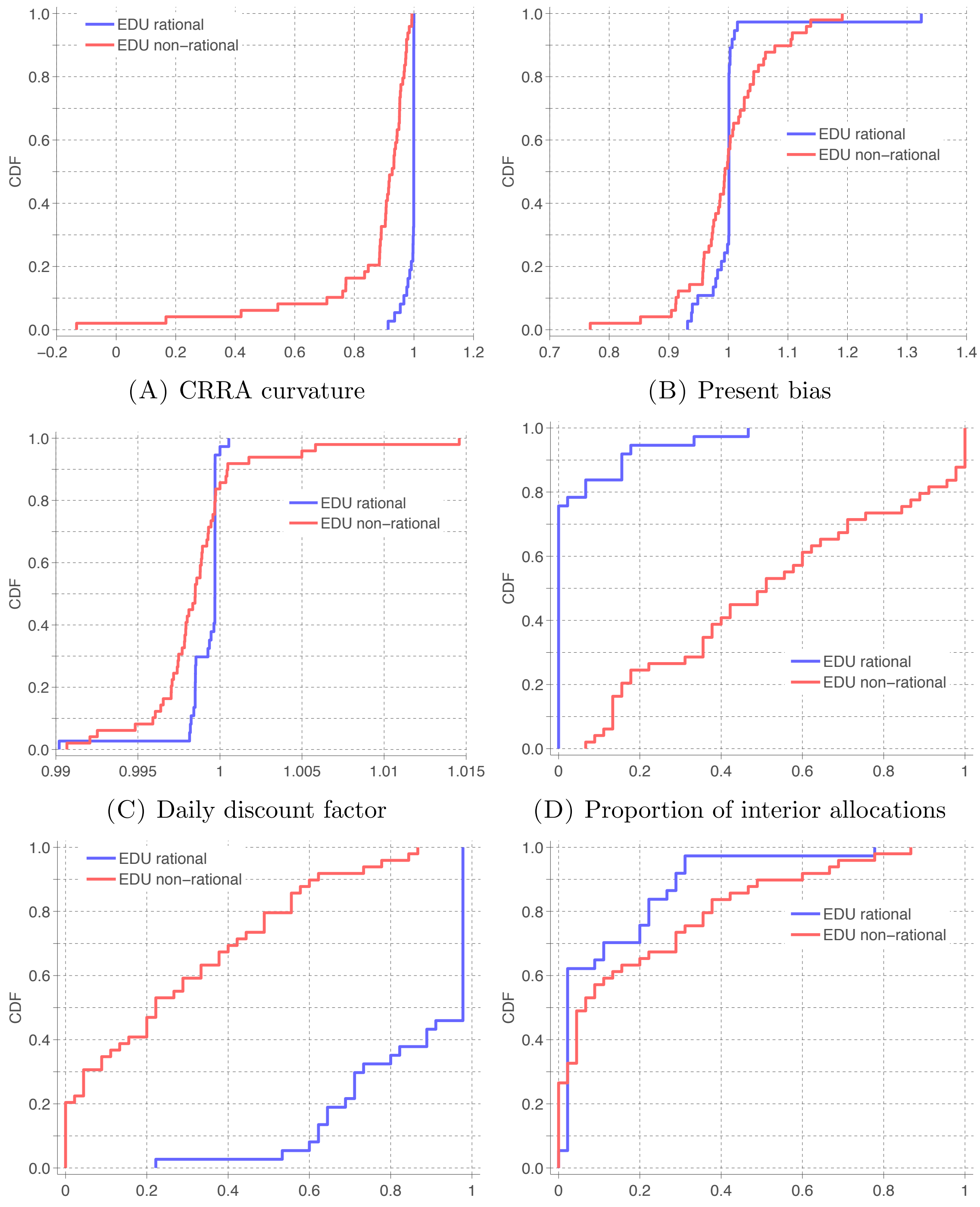
volve The situation is also illustrated in Figure 4. We classify subjects in two groups, those who violate and those who satisfy EDU. Panels (A)-(C) present empirical cumulative distribution functions (CDFs) for the estimated preference parameters in the EDU and non-EDU rational groups.

Figure 4 shows how, again, our test is clearly consistent with AS's estimates. Consider panel (B). The CDF for EDU rational subjects concentrates a large mass at $\beta = 1$. The non-EDU group has no such jump in mass at $\beta = 1$, and instead exhibits a substantial fraction of subjects with estimated β different from 1. The CDF for EDU-rational subjects is significantly different from the CDF for EDU-non-rational subjects: The null hypothesis of equality-of-distribution is rejected by the Kolmogorov-Smirnov (K-S) test ($p < 0.02$). The point is also brought out more formally by the first row of Table 1.

Figure 4 panel (B) also shows that participants who fail our EDU test have estimates of β that differ clearly from 1. A quantile regression of the absolute difference between estimated present bias and 1, $|\hat{\beta} - 1|$, on a dummy variable for EDU rationality (takes 1 if that participant fails the EDU test) reveals that the median $\hat{\beta}$ for EDU rational participants is not statistically different from 1, while median $\hat{\beta}$ for EDU non-rational participants is significantly different from 1 (Table 2, column 1). Similar quantile regression reveals that EDU non-rational participants have more heterogeneity in estimated present bias than EDU rational participants, as evident in the wider inter-quartile range of $\hat{\beta}$ (Table 2, column 3).

However, $\beta \neq 1$ is not immediately translated into evidence for present or future bias since. As we have shown above, most of the participants who fail the EDU test also fail our

[6] We obtain parameters for 86 of the 97 subjects. The remaining 11 subjects were excluded from AS's analysis, since preference parameters were not estimable. We can run our tests on the 11 excluded subjects: 9 of them pass EDU, and 10 pass QHD.



(A) CRRA curvature

(B) Present bias

(C) Daily discount factor

(D) Proportion of interior allocations

(E) Proportion of “all later” allocations

(F) Proportion of “all sooner” allocations

FIGURE 4: Empirical CDFs for preference parameters and properties of choices.

Table 1: Summary statistics of AS estimate for present bias β for each class of participants.

Class	Sample size	Percentile 5th	25th	50th	75th	95th	Inter-quartile range
EDU	37	0.938	0.999	1.001	1.001	1.015	0.0025
non EDU, TSU	18	0.768	0.971	0.993	1.003	1.078	0.0313
non TSU	31	0.904	0.959	1.003	1.042	1.138	0.0829
All sample	86	0.912	0.975	1.001	1.007	1.112	0.0318

QHD test (only 2 additional participants pass the test for P-QHD, and most of the subjects who failed EDU even fail MTD). In this sense, the interpretation of estimated β for non-EDU subjects in Figure 4 panel (B) requires some caution. The model is arguably mispecified for such subjects.

One of the advantages of our revealed preference tests is that we can go beyond the class of QHD utility function by weakening the restrictions in the relevant revealed preference axioms.

Table 1 gives us this information in a different way. The median $\hat{\beta}$'s are within 5×10^{-3} radius around 1 for all three groups, but inter-quartile range for each group is nicely ordered (see Table 2, column 4).

Consider Figure 3 again. It is interesting to note that the estimated values of $\hat{\beta}$ for participants who fail our EDU test are symmetrically distributed around 1. [7] The "average" participant looks, in some sense, as an EDU agent, even though the majority of subjects are not consistent with that model according to our test. It is therefore possible that AS's finding in favor of EDU in their aggregate preference estimation reflects the choice behavior of such an average participant.

The role of corner solutions. Next we look into subjects' choice patterns, and investigate their relationship with our test results.

For each participant, we calculate the proportions (out of 45 choices) of (i) interior allocations, (ii) corner allocations in which participants spend all her budget on a reward that is obtained later in time (called "all tokens later"), and (iii) corner allocations in which participants spend all her budget on a reward that is obtained sooner in time (called "all

[7] We test symmetry using the Kolmogorov-Smirnov test. We first sort estimated $\hat{\beta}$ in an ascending order, calculate $|\hat{\beta} - 1|$, and split them into the first half (smaller $\hat{\beta}$) and the last half (larger $\hat{\beta}$). We apply K-S test for equality of distribution for those two empirical distributions of $|\hat{\beta} - 1|$. The null hypothesis of equal distribution is not rejected ($p = 0.609$).

Table 2: Estimated present bias and class of rationality.

Independent var.	(1)	(2)	(3)	(4)
nonEDU	0.032 ***		0.063 ***	
	(0.004)		(0.016)	
$TSU\backslash EDU$		0.023 ***		0.029
		(0.002)		(0.028)
nonTSU		0.039 ***		0.080 **
		(0.002)		(0.024)
Constant	0.001	0.001	0.003	0.003
	(0.002)	(0.001)	(0.009)	(0.008)
Pseudo R^2	0.148	0.163		
0.75 pseudo R^2			0.073	0.095
0.25 pseudo R^2			0.043	0.046
# Obs.	86	86	86	86

Notes: Columns 1 and 2: Quantile regression (median) of $|\hat{\beta}-1|$ on dummy variables. Columns 3 and 4: Interquartile regression of $\hat{\beta}$ on dummy variables. *nonEDU* is a dummy for participants who fail the EDU test, $TSU\backslash EDU$ is a dummy for those who fail the EDU test but pass the TSU test, and *nonTSU* is a dummy for those who pass the TSU test. Level of significance. $*** : p < 0.01$, $** : p < 0.05$, $* : p < 0.10$.

tokens sooner").

Figure 4 panels (D)-(F) present empirical CDFs of proportions of those three types of choices for subjects who pass our EDU test, and subjects who fail our EDU test. We observe that more than 70% of the participants who pass our EDU test never made interior allocations during the experiment and frequently chose to allocate all tokens to the later payments. [8] [9] This point is made more clear in Figure 5, which presents each participant's choice pattern sorted by results of our EDU and TSU tests. The fraction of interior allocations increases by moving from EDU participants (21.7% of them made at least one interior allocation) to strictly TSU rational participants (all of them made at least one interior allocation); and it increases further when we look at subjects who fail the TSU test (all of them made at least

[8] The null hypothesis of equal distribution is rejected by K-S test for all three cases: interior allocations, $p < 10^{-14}$; "all later", $p < 10^{-13}$; "all sooner", $p < 0.04$.

[9] Andreoni and Sprenger (2012) already remark on the incidence of of corner choices, and comment on how this phenomenon may suggest that the curvature of utility is close to linear.

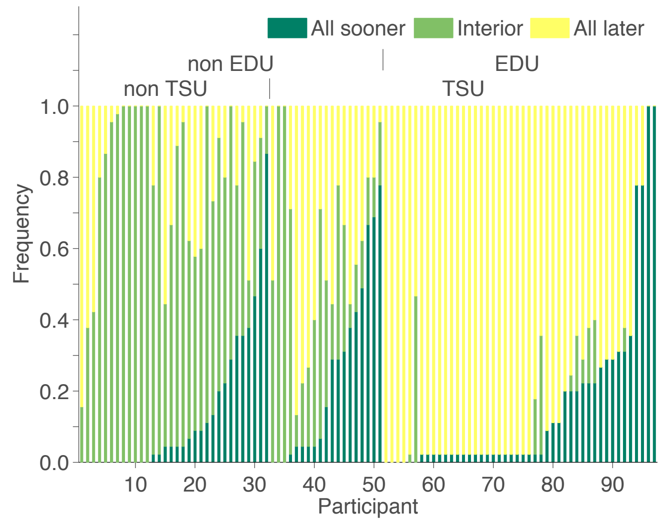


Figure 5: Individual choice patterns and class of rationality.

one interior allocation, and 15.6% of them chose interior allocations in all trials).[10][11]

The high incidence of corner solutions raises a concern. We suspect that the test for EDU rationality may have low power when one considers corner solutions. The (admittedly vague) intuition for this is that subjects' first-order conditions need to be satisfied with an inequality at a corner solution (the Kuhn-Tucker condition for optimality), while it needs to be satisfied with equality for an interior solution. We ran a simulation to assess the possibility of lower power when facing data with many corner solutions.

Assuming the same utility function in AS, we generate choice data of synthetic subjects who are present- or future- biased but have *linear utility*. Their choices therefore tend to be at the corners of the budget sets. Applying our revealed preference test to those dataset, we find *all* of the simulated observations are consistent with the revealed preference axiom for EDU.

We note that affluence of corner choices is the typical pattern observed in recent studies using the CTB method. For example, Augenblick et al. (2013) observe that 86% of monetary allocations are corner solutions and 61% of subjects have no interior allocations in twenty decisions in monetary discounting task and similarly, 31% of allocations are at corners and only 1 participant has zero interior allocations in effort discounting task. In a field setting,

[10]These observations are consistent with the fact that AS estimate for utility curvature $\hat{\alpha}$ is roughly equal to 1 (linear utility) for EDU group (Figure 4 panel (A)).

[11]We find in Figure 4 panel (B) and Figure 5 that many EDU rational participants have estimated β close to 1. Those participants made the same choice pattern: they allocated all tokens to later payment in all but one trial, in which $1 + r = 1$. They allocated all tokens to the sooner payment in that particular trial.

Table 3: Power measures.

Sampling	EDU	P-QHD	F-QHD	QHD	MTD	GTD	TSU
Uniform random	0.00	0.00	0.00	0.00	0.00	0.00	0.00
Simple Bootstrap	0.10	0.12	0.18	0.20	0.56	2.76	12.55

Giné et al. (2013) observe between 12% to 23% of corner choices under varying delay length and rate of returns. Finally, in Andreoni et al.'s (2013b) restricted CTB experiment, 87% of choices are at corners and 58% of participants have no interior allocations.

General power of the tests. It is well known that some tests in revealed preference theory tend to have low power. The low power of SARP (or GARP) is well documented. As a result, it is common (see the discussion in Andreoni et al., 2013a) to assess the power of a test by comparing the pass rates from purely random data. Here we report the results from such an assessment using our tests and the experimental design of AS. We find no evidence of low power.

We generate 10,000 datasets in which choices are made at random and uniformly distributed on the frontier of the budget set (Method 1 of Bronars, 1987). Datasets generated in this way always fail our tests (Table 3). Next, we apply the simple bootstrap method. For each of 45 budget sets, we randomly pick one choice from the set of choices observed in the entire experiment (i.e., 97 observations for each budget). We generate 10,000 such datasets and apply our revealed preference tests. We again observe high percentages of violation.

The results do not indicate a problem with power in general (but see our remarks on corner choices). The (admittedly crude) procedure of comparing pass rates of random choices do not indicate that it is easy for manifestly irrational choices to pass our tests.

Distance measure. We find that many participants' choices in the AS experiment are inconsistent with EDU, QHD, and even the TSU model. A natural question is then "how far" are these datasets from EDU, QHD, or TSU rationality. Is a participant's data not explained by EDU model because of his/her single mistake in the experiment? Or is it because of a totally inconsistent behavior?

To answer these questions we quantify the distance of the dataset from rationality by finding the largest subset of the dataset that passes the test under consideration. [12] More precisely, we take the following steps. For each EDU non-rational (similarly for QHD and

[12]This approach is motivated by a measure proposed by Houtman and Maks (1985), which is based on finding the largest subset of observations that is consistent with GARP.

TSU) participant's dataset: (i) We randomly drop one observation from the dataset; (ii) We implement the EDU test. If the dataset is EDU rational, we stop here. Otherwise, we drop another observation randomly and test EDU rationality again; and (iii) We repeat this procedure until the subset becomes EDU rational.

Ideally, one would consider dropping all possible subsets of data, but such a calculation is obviously computationally infeasible. Our approach of sequentially choosing (at random) one observation to drop is a rough approximation of the ideal measure. In particular, the conclusion can depend on the particular sequence chosen. To address this problem we iterate the process $1,000$ times for each EDU non-rational participant. Let n_m be the number of observations required to be dropped from the original dataset to make the subdata EDU rational, in the m-th iteration. We define the distance of the dataset from EDU rationality by $d'_{\text{EDU}} = \min\{n_1, \ldots, n_{1000}\}/45$. By definition, the measure is between 0 and 1, and the smaller d'_{EDU} is the closer the dataset to be EDU rational. We also note that the measure is an "upper bound" of the distance we want to capture, due to random nature and path-dependence of our approach. [13]

Figure 6 shows empirical CDFs of d'_{EDU} along with d'_{QHD} and d'_{TSU}, calculated in the similar manner. Note that sample size is different for each line: d'_{EDU} is calculated for 51 EDU non-rational participants, d'_{QHD} is calculated for 49 QHD non-rational participants, and d'_{EDU} is calculated for 32 TSU non-rational participants. We find that the median d'_{TSU} is 0.089. implying that half of the 32 TSU non-rational participants become TSU rational by dropping 9% of the observations. For EDU and QHD, on the other hand, more observations need to be dropped to rationalize the data: median d'_{EDU} and d'_{QHD} are 0.356 and 0.333, respectively. We also find that the distributions of d'_{EDU} and d'_{QHD} are almost indistinguishable.

In Figure 7 we see a significant positive correlation (Pearson's correlation coefficient $\rho = 0.7687$, $p < 10^{-10}$) between the proportion of interior allocations and the distance to EDU rationality. This correlation is in line with our speculation that the affluence of corner allocations make our revealed preference tests less demanding.

[13] We should observe $d'_{\text{EDU}} \geq d'_{\text{QHD}} \geq d'_{\text{TSU}}$ as a logical consequence (if the subset of data, after dropping n observations, is EDU rational, then the same subset is QHD rational, and so on). In reality, however, due to sample variation in the stochastic algorithm we use to compute distances, we observe several instances in which $d'_{\text{EDU}} < d'_{\text{QHD}}$ (or $d'_{\text{EDU}} < d'_{\text{TSU}}$). We correct for this by simply replacing d'_{QHD} with d'_{EDU} (similarly, d'_{TSU} with d'_{EDU}) whenever such violations are observed.

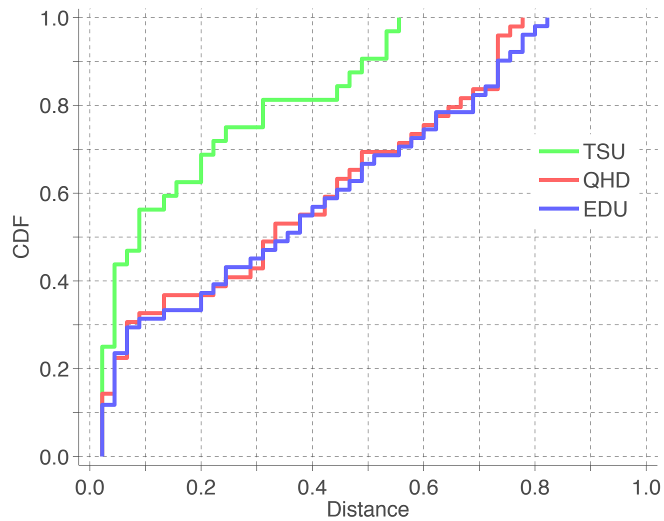


FIGURE 6: Empirical CDFs of distance measures of dataset from EDU, QHD, and TSU rationality.

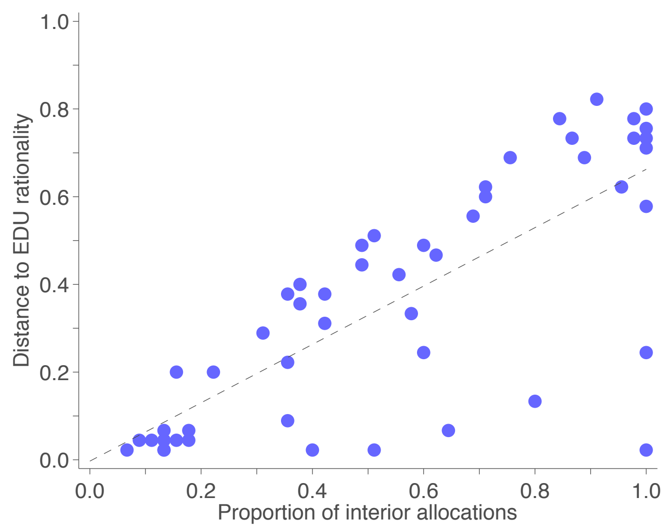


FIGURE 7: Distance to EDU rationality and proportion of interior allocations.

7 Proof of Theorem 1

We present the proof of the equivalence between EDU rationality and SA-EDU rationality. Except for the equivalence between TSU and SA-TSU, the proof of the remaining claims in Theorem 1 is similar. We comment on how the proof for EDU needs to be modified to obtain each of the remaining claims.

8 Proof of Theorem 1 (i)

The proof is based on using the first-order conditions for maximizing a utility with the EDU model over a budget set. Our first lemma ensures that we can without loss of generality restrict attention to first order conditions. The proof of the lemma is the same as that of Lemma 3 in Echenique and Saito (2013a) with the changes of T to S and $\{\delta^t\}_{t\in T}$ to $\{\mu_s\}_{s\in S}$.

We use the following notation in the proofs:

$$\mathcal{X} = \{x_t^k : k = 1, \ldots, K, t = 0, \ldots, T\}.$$

Lemma 1. *Let $(x^k, p^k)_{k=1}^K$ be a dataset. The following statements are equivalent:*

1. *$(x^k, p^k)_{k=1}^K$ is EDU rational.*
2. *There are strictly positive numbers v_t^k, λ^k, and $\delta \in (0, 1]$, for $t = 1, \ldots, T$ and $k = 1, \ldots, K$, such that*

$$\delta^t v_t^k = \lambda^k p_t^k$$
$$x_t^k > x_{t'}^{k'} \Rightarrow v_t^k \leq v_{t'}^{k'}.$$

Proof. We shall prove that (1) implies (2). Let $(x^k, p^k)_{k=1}^K$ be EDU rational. Let $\delta \in (0, 1]$ and $u : \mathbf{R}_+ \to \mathbf{R}$ be as in the definition of EDU rational data. Then (see, for example, Theorem 28.3 of Rockafellar (1997)), there are numbers $\lambda^k \geq 0$, $k = 1, \ldots, K$ such that if we let

$$v_t^k = \frac{\lambda^k p_t^k}{\delta^t}$$

then $v_t^k \in \partial u(x_t^k)$ if $x_t^k > 0$, and there is $\underline{w} \in \partial u(x_t^k)$ with $v_t^k \geq \underline{w}$ if $x_t^k = 0$. In fact, it is easy to see that $\lambda^k > 0$, and therefore $v_t^k > 0$.

By the concavity of u, and the consequent monotonicity of $\partial u(x_t^k)$ (see Theorem 24.8 of Rockafellar (1997)), if $x_t^k > x_{t'}^{k'} > 0$, $v_t^k \in \partial u(x_t^k)$, and $v_{t'}^{k'} \in \partial u(x_{t'}^{k'})$, then $v_t^k \leq v_{t'}^{k'}$. If $x_t^k > x_{t'}^{k'} = 0$, then $\underline{w} \in \partial u(x_{t'}^{k'})$ with $v_{t'}^{k'} \geq \underline{w}$. So $v_t^k \leq \underline{w} \leq v_{t'}^{k'}$.

In second place, we show that (2) implies (1). Suppose that the numbers v_t^k, λ^k, δ, for $t \in T$ and $k \in K$, are as in (2).

Enumerate the elements in $\mathcal{X}$ in increasing order:

$$y_1 < y_2 < \ldots < y_n$$

Let

$$\underline{y}_i = \min\{v_t^k : x_t^k = y_i\} \text{ and } \bar{y}_i = \max\{v_t^k : x_t^k = y_i\}.$$

Let $z_i = (y_i + y_{i+1})/2$, $i = 1, \ldots, n-1$; $z_0 = 0$, and $z_n = y_n + 1$. Let f be a correspondence defined as follows:

$$f(z) = \begin{cases} [\underline{y}_i, \bar{y}_i] & \text{if } z = y_i, \\ \max\{\bar{y}_i : z < y_i\} & \text{if } y_n > z \text{ and } \forall i (z \neq y_i), \\ \underline{y}_n/2 & \text{if } y_n < z. \end{cases}$$

By assumption of the numbers v_t^k, we have that, when $y < y'$, $v \in f(y)$ and $v' \in f(y')$, then $v \leq v'$. Then the correspondence f is monotone and there is a concave function u for which $\partial u = f$ (Theorem 24.8 of Rockafellar (1997)). Given that $v_t^k > 0$ all the elements in the range of f are positive, and therefore u is strictly increasing.

Finally, for all (k,t), $\lambda^k p_t^k / \delta^t = v_t^k \in \partial u(v_t^k)$ and therefore the first-order conditions to a maximum choice of x hold at x_t^k. Since u is concave the first-order conditions are sufficient. The data is therefore EDU rational. □

8.1 Necessity

Lemma 2. *If a dataset $(x^k, p^k)_{k=1}^K$ is EDU rational, then it satisfies SA-EDU.*

Proof. Let $(x^k, p^k)_{k=1}^K$ be EDU rational, and let $\delta \in (0,1]$ and $u : \mathbf{R}_+ \to \mathbf{R}$ be as in the definition of EDU rational. By Lemma 1, there exists a strictly positive solution v_t^k, λ^k, δ to the system in Statement (2) of Lemma 1 with $v_t^k \in \partial u(x_t^k)$ when $x_t^k > 0$, and $v_t^k \geq \underline{w} \in \partial u(x_t^k)$ when $x_t^k = 0$.

Let $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^n$ be a sequence satisfying the three conditions in SA-EDU. Then $x_{t_i}^{k_i} > x_{t_i'}^{k_i'}$. Suppose that $x_{t_i'}^{k_i'} > 0$. Then, $v_{t_i}^{k_i} \in \partial u(x_{t_i}^{k_i})$ and $v_{t_i'}^{k_i'} \in \partial u(x_{t_i'}^{k_i'})$. By the concavity of u, it follows that $\lambda^{k_i} \delta^{t_i'} p_{t_i}^{k_i} \leq \lambda^{k_i'} \delta^{t_i} p_{t_i'}^{k_i'}$ (see Theorem 24.8 of Rockafellar (1997)). Similarly, if $x_{t_i'}^{k_i'} = 0$, then $v_{t_i}^{k_i} \in \partial u(x_{t_i}^{k_i})$ and $v_{t_i'}^{k_i'} \geq \underline{w} \in \partial u(x_{t_i'}^{k_i'})$. So $\lambda^{k_i} \delta^{t_i'} p_{t_i}^{k_i} \leq \lambda^{k_i'} \delta^{t_i} p_{t_i'}^{k_i'}$ Therefore,

$$1 \geq \prod_{i=1}^n \frac{\lambda^{k_i} \delta^{t_i'} p_{t_i}^{k_i}}{\lambda^{k_i'} \delta^{t_i} p_{t_i'}^{k_i'}} = \frac{1}{\delta^{(\sum t_i - \sum t_i')}} \prod_{i=1}^n \frac{p_{t_i}^{k_i}}{p_{t_i'}^{k_i'}} \geq \prod_{i=1}^n \frac{p_{t_i}^{k_i}}{p_{t_i'}^{k_i'}},$$

as the sequence satisfies (2) and (3) of SA-EDU; and hence $\sum t_i \geq \sum t'_i$ and the numbers λ^k appear the same number of times in the denominator as in the numerator of this product. $\square$

8.2 Theorem of Alternatives

To prove the sufficiency, we shall use the following lemma, which is a version of the Theorem of the Alternative. This is Theorem 1.6.1 in Stoer and Witzgall (1970). We shall use it here in the cases where F is either the real or the rational numbers.

Lemma 3. *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E belong the a commutative ordered field $\mathbf{F}$. Exactly one of the following alternatives is true.*

1. *There is $u \in \mathbf{F}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, $E \cdot u \gg 0$.*

2. *There is $\theta \in \mathbf{F}^r$, $\eta \in \mathbf{F}^l$, and $\pi \in \mathbf{F}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$.*

We also use the following lemma, which follows from Lemma 3 (See Border (2013) or Chambers and Echenique (2014)):

Lemma 4. *Let A be an $m \times n$ matrix, B be an $l \times n$ matrix, and E be an $r \times n$ matrix. Suppose that the entries of the matrices A, B, and E are rational numbers. Exactly one of the following alternatives is true.*

1. *There is $u \in \mathbf{R}^n$ such that $A \cdot u = 0$, $B \cdot u \geq 0$, and $E \cdot u \gg 0$.*

2. *There is $\theta \in \mathbf{Q}^r$, $\eta \in \mathbf{Q}^l$, and $\pi \in \mathbf{Q}^m$ such that $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$; $\pi > 0$ and $\eta \geq 0$.*

8.3 Sufficiency

We proceed to prove the sufficiency direction. Sufficiency follows from the following lemmas as in Echenique and Saito (2013a).

We know from Lemma 1 that it suffices to find a solution to the first order conditions. Lemma 5 establishes that SA-EDU is sufficient when the logarithms of the prices are rational numbers. The role of rational logarithms comes from our use of a version of Farkas's Lemma. Lemma 6 says that we can approximate any data satisfying SA-EDU with a dataset for which the logs of prices are rational and for which SA-EDU is satisfied. Finally, Lemma 7 establishes the result. It is worth mentioning that we cannot use Lemma 6 and an approximate solution to obtain a limiting solution.

Lemma 5. *Let data $(x^k, p^k)_{k=1}^k$ satisfy SA-EDU. Suppose that $\log(p_t^k) \in \mathbf{Q}$ for all k and t. Then there are numbers v_t^k, λ^k, δ, for $t \in T$ and $k = 1, \ldots, K$ satisfying* (2) *in Lemma 1.*

Lemma 6. *Let data $(x^k, p^k)_{k=1}^k$ satisfy SA-EDU. Then for all positive numbers $\overline{\varepsilon}$, there exists $q_t^k \in [p_t^k - \overline{\varepsilon}, p_t^k]$ for all $t \in T$ and $k \in K$ such that $\log q_t^k \in \mathbf{Q}$ and the dataset $(x^k, q^k)_{k=1}^k$ satisfy SA-EDU.*

Lemma 7. *Let data $(x^k, p^k)_{k=1}^k$ satisfy SA-EDU. Then there are numbers v_t^k, λ^k, δ, for $t \in T$ and $k = 1, \ldots, K$ satisfying* (2) *in Lemma 1.*

8.4 Proof of Lemma 5

We linearize the equation in System (2) of Lemma 1. The result is:

$$\log v(x_t^k) + t \log \delta - \log \lambda^k - \log p_t^k = 0, \tag{2}$$

$$x > x' \Rightarrow \log v(x') \geq \log v(x), \tag{3}$$

$$\log \delta \leq 0. \tag{4}$$

In the system comprised by (2), (3), and (4), the unknowns are the real numbers $\log v_t^k$, $\log \delta$, $k \in K$ and $t \in T$.

First, we are going to write the system of inequalities (2) and (3) in matrix form.

A system of linear inequalities

We shall define a matrix A such that there are positive numbers v_t^k, λ^k, δ the logs of which satisfy Equation (2) if and only if there is a solution $u \in \mathbf{R}^{K\times(T+1)+1+K+1}$ to the system of equations

$$A \cdot u = 0,$$

and for which the last component of u is strictly positive.

Let A be a matrix with $K \times (T + 1) + 1 + K + 1$ columns, defined as follows: We have one row for every pair (k, t) such that $x_t^k > 0$; one column for every pair (k, t); one column for each k; and two additional columns. Organize the columns so that we first have the $K \times (T + 1)$ columns for the pairs (k, t); then one of the single columns mentioned in last place, which we shall refer to as the δ-column; then K columns (one for each k); and finally one last column. In the row corresponding to (k, t) the matrix has zeroes everywhere with the following exceptions: it has a 1 in the column for (k, t); it has a t in the δ column; it has a -1 in the column for k; and $-\log p_t^k$ in the very last column.

Thus, matrix A looks as follows:

$$\begin{array}{c} \\ \vdots \\ (k,t) \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{ccccc|c|ccccc|c}
{\scriptstyle (1,0)} & \cdots & {\scriptstyle (k,t)} & \cdots & {\scriptstyle (K,T)} & {\scriptstyle \delta} & {\scriptstyle 1} & \cdots & {\scriptstyle k} & \cdots & {\scriptstyle K} & {\scriptstyle p}
\end{array} \\
\left[\begin{array}{ccccc|c|ccccc|c}
\vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 1 & \cdots & 0 & t & 0 & \cdots & -1 & \cdots & 0 & -\log p_t^k \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots
\end{array}\right]
\end{array}$$

Consider the system $A \cdot u = 0$. If there are numbers solving Equation (2), then these define a solution $u \in \mathbf{R}^{K\times(T+1)+1+K+1}$ for which the last component is 1. If, on the other hand, there is a solution $u \in \mathbf{R}^{K\times(T+1)+1+K+1}$ to the system $A \cdot u = 0$ in which the last component is strictly positive, then by dividing through by the last component of u we obtain numbers that solve Equation (2).

In second place, we write the system of inequalities (3) in matrix form. Let B be a matrix B with $K \times (T+1) + 1 + K + 1$ columns. Define B as follows: One row for every pair (k,t) and (k',t') with $x_t^k > x_{t'}^{k'}$; in the row corresponding to (k,t) and (k',t') we have zeroes everywhere with the exception of a -1 in the column for (k,t) and a 1 in the column for (k',t') . Finally, in the last row, we have zero everywhere with the exception of a -1 at $K \times (T+1) + 1$th column. We shall refer to this last row as the δ-row.

In third place, we have a matrix E that captures the requirement that the last component of a solution be strictly positive. The matrix E has a single row and $K \times (T+1)+1+K+1$ columns. It has zeroes everywhere except for 1 in the last column.

To sum up, there is a solution to system (2), (3) and (4) if and only if there is a vector $u \in \mathbf{R}^{K\times(T+1)+1+K+1}$ that solves the system of equations and linear inequalities

$$S1: \begin{cases} A \cdot u = 0, \\ B \cdot u \geq 0, \\ E \cdot u \gg 0. \end{cases}$$

Theorem of the Alternative

The entries of A, B, and E are integer numbers, with the exception of the last column of A. Under the hypothesis of the lemma we are proving, the last column consists of rational numbers.

By Lemma 4, then, there is such a solution u to $S1$ if and only if there is no rational vector (θ, η, π) that solves the system of equations and linear inequalities

$$S2: \begin{cases} \theta \cdot A + \eta \cdot B + \pi \cdot E = 0, \\ \eta \geq 0, \\ \pi > 0. \end{cases}$$

In the following, we shall prove that the non-existence of a solution u implies that the data must violate SA-EDU. Suppose then that there is no solution u and let (θ, η, π) be a rational vector as above, solving system $S2$.

By multiplying (θ, η, π) by any positive integer we obtain new vectors that solve $S2$, so we can take (θ, η, π) to be integer vectors.

Henceforth, we use the following notational convention: For a matrix D with $K \times (T + 1) + 1 + K + 1$ columns, write D_1 for the submatrix of D corresponding to the first $K \times (T + 1)$ columns; let D_2 be the submatrix corresponding to the following one column (i.e., δ-column); D_3 correspond to the next K columns; and D_4 to the last column. Thus, $D = [D_1|D_2|D_3|D_4]$.

Claim 1. *(i)* $\theta \cdot A_1 + \eta \cdot B_1 = 0$*; (ii)* $\theta \cdot A_2 + \eta \cdot B_2 = 0$*; (iii)* $\theta \cdot A_3 = 0$*; and (iv)* $\theta \cdot A_4 + \pi \cdot E_4 = 0$.

Proof. Since $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$, then $\theta \cdot A_i + \eta \cdot B_i + \pi \cdot E_i = 0$ for all $i = 1, \ldots, 4$. Moreover, since B_3, B_4, E_1, E_2, and E_3 are zero matrices, we obtain the claim. □

For convenience, we transform the matrices A and B using θ and η.

Transform the matrices A, and B

Lets define a matrix A^* from A by letting A^* have $K \times (T + 1) + 1 + K + 1$ columns that consists of the rows as follows: for each row in $r \in A$

1. θ_r copies of the rth row when $\theta_r > 0$;

2. omitting row r when $\theta_r = 0$;

3. θ_r copies of the rth row multiplied by -1 when $\theta_r < 0$.

We refer to rows that are copies of some r in A with $\theta_r > 0$ as *original* rows. We refer to rows that are copies of some r in A with $\theta_r < 0$ as *converted* rows.

Similarly, we define the matrix B^* from B by including the same columns as B and η_r copies of each row (and thus omitting row r when $\eta_r = 0$; recall that $\eta_r \geq 0$ for all r).

Claim 2. *For any* (k, t)*, all the entries in the column for* (k, t) *in* A^*_1 *are of the same sign.*

Proof. By definition of A, the column for (k, t) will have zero in all its entries with the exception of the row for (k, t). In A^*, for each (k, t), there are three mutually exclusive possibilities: the row for (k, t) in A can (i) not appear in A^*, (ii) it can appear as original, or (iii) it can appear as converted. This shows the claim. □

Claim 3. *There exists a sequence of pairs* $(x^{k_i}_{t_i}, x^{k'_i}_{t'_i})_{i=1}^{n^*}$ *that satisfies (1) in SA-EDU.*

Proof. We define such a sequence by induction. Let $B^1 = B^*$. Given B^i, define B^{i+1} as follows.

Denote by $>^i$ the binary relation on $\mathcal{X}$ defined by $z >^i z'$ if $z > z'$ and there is at least one pair (k,t) and (k',t') for which (i) $x^k_t > x^{k'}_{t'}$; (ii) $z = x^k_t$ and $z' = x^{k'}_{t'}$; and (iii) the row corresponding $x^k_t > x^{k'}_{t'}$ in B has strictly positive weight in η.

The binary relation $>^i$ cannot exhibit cycles because $>^i \subseteq >$. There is therefore at least one sequence $z^i_1, \ldots z^i_{L_i}$ in $\mathcal{X}$ such that $z^i_j >^i z^i_{j+1}$ for all $j = 1, \ldots, L_i - 1$ and with the property that there is no $z \in \mathcal{X}$ with $z >^i z^i_1$ or $z^i_{L_i} >^i z$.

Let the matrix B^{i+1} be defined as the matrix obtained from B^i by omitting one copy of the row corresponding to $z^i_j > z^i_{j+1}$, for all $j = 1, \ldots L_i - 1$.

The matrix B^{i+1} has strictly fewer rows than B^i. There is therefore n^* for which B^{n^*+1} either has no more rows, or $B^{n^*+1}_1$ has only zeroes in all its entries (its rows are copies of the δ-row which has only zeroes in its first $K \times (T+1)$ columns).

Define a sequence of pairs $(x^{k_i}_{t_i}, x^{k'_i}_{t'_i})_{i=1}^{n^*}$ by letting $x^{k_i}_{t_i} = z^i_1$ and $x^{k'_i}_{t'_i} = z^i_{L_i}$. Note that, as a result, $x^{k_i}_{t_i} > x^{k'_i}_{t'_i}$ for all i. Therefore the sequence of pairs $(x^{k_i}_{t_i}, x^{k'_i}_{t'_i})_{i=1}^{n^*}$ satisfies condition (1) in SA-EDU. □

We shall use the sequence of pairs $(x^{k_i}_{t_i}, x^{k'_i}_{t'_i})_{i=1}^{n^*}$ as our candidate violation of SA-EDU.

Consider a sequence of matrices A^i, $i = 1, \ldots, n^*$ defined as follows. Let $A^1 = A^*$, $B^1 = B^*$, and

$$C^1 = \begin{bmatrix} A^1 \\ B^1 \end{bmatrix}.$$

Observe that the rows of C^1 add to the null vector by Claim 9.

We shall proceed by induction. Suppose that A^i has been defined, and that the rows of

$$C^i = \begin{bmatrix} A^i \\ B^i \end{bmatrix}$$

add to the null vector.

Recall the definition of the sequence

$$x^{k_i}_{t_i} = z^i_1 > \ldots > z^i_{L_i} = x^{k'_i}_{t'_i}.$$

There is no $z \in \mathcal{X}$ with $z >^i z^i_1$ or $z^i_{L_i} >^i z$, so in order for the rows of C^i to add to zero there must be a -1 in A^i_1 in the column corresponding to (k'_i, t'_i) and a 1 in A^i_1 in the column corresponding to (k_i, t_i). Let r_i be a row in A^i corresponding to (k_i, t_i), and r'_i be a row corresponding to (k_i', t_i'). The existence of a -1 in A_1^i in the column corresponding to (k_i', t_i'), and a 1 in A_1^i in the column corresponding to (k_i, t_i), ensures that r_i and r_i' exist. Note that the row r_i' is a converted row while r_i is original. Let A^{i+1} be defined from A^i by deleting the two rows, r_i and r_i'.

Claim 4. *The sum of r_i, r_i', and the rows of B^i which are deleted when forming B^{i+1} (corresponding to the pairs $z_j^i > z_{j+1}^i$, $j = 1, \dots, L_i - 1$) add to the null vector.*

Proof. Recall that $z_j^i >^i z_{j+1}^i$ for all $j = 1, \dots, L_i - 1$. So when we add the rows corresponding to $z_j^i >^i z_{j+1}^i$ and $z_{j+1}^i >^i z_{j+2}^i$, then the entries in the column for (k, t) with $x_t^k = z_{j+1}^i$ cancel out and the sum is zero in that entry. Thus, when we add the rows of B^i that are not in B^{i+1} we obtain a vector that is 0 everywhere except the columns corresponding to z_1^i and $z_{L_i}^i$. This vector cancels out with $r_i + r_i'$, by definition of r_i and r_i'. □

Claim 5. *The matrix A^* can be partitioned into pairs of rows as follows:*

$$A^* = \begin{bmatrix} r_1 \\ r_1' \\ \vdots \\ r_i \\ r_i' \\ \vdots \\ r_{n^*} \\ r_{n^*}', \end{bmatrix}$$

in which the rows r_i' are converted and the rows r_i are original.

Proof. For each i, A^{i+1} differs from A^i in that the rows r_i and r_i' are removed from A^i to form A^{i+1}. We shall prove that A^* is composed of the $2n^*$ rows r_i, r_i'.

First note that since the rows of C^i add up to the null vector, and A^{i+1} and B^{i+1} are obtained from A^i and B^i by removing a collection of rows that add up to zero, then the rows of C^{i+1} must add up to zero as well.

By way of contradiction, suppose that there exist rows left after removing r_{n^*} and r_{n^*}'. Then, by the argument above, the rows of the matrix C^{n^*+1} must add to the null vector. If there are rows left, then the matrix C^{n^*+1} is well defined.

By definition of the sequence B^i, however, B^{n^*+1} has all its entries equal to zero, or has no rows. Hence, the rows remaining in $A_1^{n^*+1}$ must add up to zero. By Claim 10, the entries of a column (k,t) of A^* are always of the same sign. Moreover, each row of A^* has a non-zero element in the first $K \times S$ columns. Therefore, no subset of the columns of A_1^* can sum to the null vector. $\square$

Claim 6. *(i) For any k and t, if $(k_i, t_i) = (k,t)$ for some i, then the row r_i corresponding to (k,t) appears as original in A^*. Similarly, if $(k_i', t_i') = (k', t')$ for some i, then the row corresponding to (k,t) appears converted in A^*.*
(ii) If the row corresponding to (k,t) appears as original in A^*, then there is some i with $(k_i, t_i) = (k,t)$. Similarly, if the row corresponding to (k,t) appears converted in A^*, then there is i with $(k_i', t_i') = (k,t)$.

Proof. (i) is true by definition of $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})$. (ii) is immediate from Claim 13 because if the row corresponding to (k,t) appears original in A^* then it equals r_i for some i, and then $x_t^k = x_{t_i}^{k_i}$. Similarly when the row appears converted. $\square$

Claim 7. *The sequence $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^{n^*}$ satisfies (2) and (3) in SA-EDU.*

Proof. We first establish (2). Note that A_2^* is a vector, and in row r the entry of A_2^* is as follows. There must be a (k,t) of which r is a copy. Then the component at row r of A_2^* is t if r is original and $-t$ if r is converted. Now, when r appears as original there is some i for which $t = t_i$, when r appears as converted there is some i for which $t = t_i'$. So for each r there is i such that $(A_4^*)_r$ is either t_i or $-t_i'$. By Claim 9 (ii), $\theta \cdot A_2 + \eta \cdot B_2 = 0$. Recall that $\theta \cdot A_2$ equals the sum of the rows of A_2^*. Moreover, B_2 is a vector that has zeroes everywhere except a -1 in the δ row (i.e., $K \times (T+1) + 1$th row). Therefore, the sum of the rows of A_2^* equals $\eta_{K\times(T+1)+1}$, where $\eta_{K\times(T+1)+1}$ is the $K \times (T+1) + 1$th element of η. Since $\eta \geq 0$, therefore, $\sum_{i=1}^{n^*} t_i \geq \sum_{i=1}^{n^*} t_i'$, and condition (2) in the axiom is satisfied.

Now we turn to (3). By Claim 9 (iii), the rows of A_3^* add up to zero. Therefore, the number of times that k appears in an original row equals the number of times that it appears in a converted row. By Claim 14, then, the number of times k appears as k_i equals the number of times it appears as k_i'. Therefore condition (3) in the axiom is satisfied. $\square$

Finally, in the following, we show that

$$\prod_{i=1}^{n^*} \frac{p_{t_i}^{k_i}}{p_{t_i'}^{k_i'}} > 1,$$

which finishes the proof of Lemma 5 as the sequence $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^{n^*}$ would then exhibit a violation of SA-EDU.

Claim 8. $\prod_{i=1}^{n^*} \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}} > 1$.

Proof. By Claim 9 (iv) and the fact that the submatrix E_4 equals the scalar 1, we obtain

$$0 = \theta \cdot A_4 + \pi E_4 = (\sum_{i=1}^{n^*}(r_i + r'_i))_4 + \pi,$$

where $(\sum_{i=1}^{n^*}(r_i+r'_i))_4$ is the (scalar) sum of the entries of A_4^*. Recall that $-\log p_{t_i}^{k_i}$ is the last entry of row r_i and that $\log p_{t'_i}^{k'_i}$ is the last entry of row r'_i, as r'_i is converted and r_i original. Therefore the sum of the rows of A_4^* are $\sum_{i=1}^{n^*} \log(p_{t'_i}^{k'_i}/p_{t_i}^{k_i})$. Then,

$$\sum_{i=1}^{n^*} \log(p_{t'_i}^{k'_i}/p_{t_i}^{k_i}) = -\pi < 0.$$

Thus

$$\prod_{i=1}^{n^*} \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}} > 1.$$

□

8.5 Proof of Lemma 6

For each sequence $\sigma = (x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^n$, we shall identify the pair $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})$ with $((k_i, t_i), (k'_i, t'_i))$.

For each sequence $\sigma = (x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^n$ that satisfies conditions (1), (2), and (3) in SA-EDU, we define a vector $t_\sigma \in \mathbf{N}^{K^2T^2}$. Let $t_\sigma((k,t),(k',t'))$ be the number of times that the pair $(x_t^k, x_{t'}^{k'})$ appears in the sequence σ. One can then describe the satisfaction of SA-EDU by means of the vectors t_σ. Define

$$T \quad = \quad \Big\{ t_\sigma \in \mathbf{N}^{K^2T^2} | \sigma \text{ satisfies (1), (2), (3) in SA-EDU } \Big\}.$$

Observe that the set T depends only on $(x^k)_{k=1}^K$ in the dataset $(x^k, p^k)_{k=1}^K$. It does not depend on prices.

For each $((k,t),(k',t'))$ such that $x_t^k > x_{t'}^{k'}$, define

$$\hat{\beta}((k,t),(k',t')) = \log \left(\frac{p_t^k}{p_{t'}^{k'}} \right).$$

And define $\hat{\beta}((k,t),(k',t')) = 0$ when $x_t^k \leq x_{t'}^{k'}$. Then, $\hat{\beta}$ is a K^2T^2-dimensional real-valued vector.

If $\sigma = (x^{k_i}_{t_i}, x^{k'_i}_{t'_i})^n_{i=1}$, then

$$\hat{\beta} \cdot t_\sigma = \sum_{((k,t),(k',t'))} \hat{\beta}((k,t),(k',t'))t_\sigma((k,t),(k',t')) = \log\Big(\prod_{i=1}^n \frac{p^{k_i}_{t_i}}{p^{k'_i}_{t'_i}}\Big).$$

So the data satisfy SA-EDU if and only if $t \cdot \hat{\gamma} \leq 0$ for all $t \in T$.

Enumerate the elements in $\mathcal{X}$ in increasing order:

$$y_1 < y_2 < \cdots < y_N.$$

Fix an arbitrary $\underline{\xi} \in (0,1)$.

We shall construct by induction a sequence $(\varepsilon^k_t(n))$ for $n = 1, \ldots, N$, where $\varepsilon^k_t(n)$ is defined for all (k,t) with $x^k_t = y_n$.

By the denseness of the rational numbers, and the continuity of the exponential function, for each (k,t) such that $x^k_t = y_1$, there exists a positive number $\varepsilon^k_t(1)$ such that $\log(p^k_t\varepsilon^k_t(1)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon^k_t(1) < 1$. Let $\varepsilon(1) = \min\{\varepsilon^k_t(1) | x^k_t = y_1\}$.

In second place, for each (k,t) such that $x^k_t = y_2$, there exists a positive $\varepsilon^k_t(2)$ such that $\log(p^k_t\varepsilon^k_t(2)) \in \mathbf{Q}$ and $\underline{\xi} < \varepsilon^k_t(2) < \varepsilon(1)$. Let $\varepsilon(2) = \min\{\varepsilon^k_t(2) | x^k_t = y_2\}$.

In third place, and reasoning by induction, suppose that $\varepsilon(n)$ has been defined and that $\underline{\xi} < \varepsilon(n)$. For each (k,t) such that $x^k_t = y_{n+1}$, let $\varepsilon^k_t(n+1) > 0$ be such that $\log(p^k_t\varepsilon^k_t(n+1)) \in \mathbf{Q}$, and $\underline{\xi} < \varepsilon^k_t(n+1) < \varepsilon(n)$. Let $\varepsilon(n+1) = \min\{\varepsilon^k_t(n+1) | x^k_t = y_n\}$.

This defines the sequence $(\varepsilon^k_t(n))$ by induction. Note that $\varepsilon^k_t(n+1)/\varepsilon(n) < 1$ for all n. Let $\bar{\xi} < 1$ be such that $\varepsilon^k_t(n+1)/\varepsilon(n) < \bar{\xi}$.

For each $k \in K$ and $t \in T$, let $q^k_t = p^k_t\varepsilon^k_t(n)$, where n is such that $x^k_t = y_n$. We claim that the data $(x^k, q^k)^K_{k=1}$ satisfy SA-EDU. Let γ^* be defined from $(q^k)^K_{k=1}$ in the same manner as $\hat{\gamma}$ was defined from $(p^k)^K_{k=1}$.

For each pair $((k,t),(k',t'))$ with $x^k_t > x^{k'}_{t'}$, if n and m are such that $x^k_t = y_n$ and $x^{k'}_{t'} = y_m$, then $n > m$. By definition of ε,

$$\frac{\varepsilon^k_t(n)}{\varepsilon^{k'}_{t'}(m)} < \frac{\varepsilon^k_t(n)}{\varepsilon(m)} < \bar{\xi} < 1.$$

Hence,

$$\gamma^*((k,t),(k',t')) = \log\frac{p^k_t\varepsilon^k_t(n)}{p^{k'}_{t'}\varepsilon^{k'}_{t'}(m)} < \log\frac{p^k_t}{p^{k'}_{t'}} + \log\bar{\xi} < \log\frac{p^k_t}{p^{k'}_{t'}} = \hat{\gamma}(x^k_s, x^{k'}_{t'}).$$

Thus, for all $t \in T$,

$$\gamma^* \cdot t \leq \hat{\gamma} \cdot t \leq 0,$$

as $t \geq 0$ and the data $(x^k, p^k)^K_{k=1}$ satisfy SA-EDU. Thus the data $(x^k, q^k)^K_{k=1}$ satisfy SA-EDU.

Finally, note that $\underline{\xi} < \varepsilon^k_t(n) < 1$ for all n and each $k \in K, t \in T$. So that by choosing $\underline{\xi}$ close enough to 1 we can take the prices (q^k) to be as close to (p^k) as desired.

8.6 Proof of Lemma 7

Consider the system comprised by (2) and (3) in the proof of Lemma 5. Let A, B, and E be constructed from the data as in the proof of Lemma 5. The difference with respect to Lemma 5 is that now the entries of A_4 may not be rational. Note that the entries of E, B, and A_i, $i = 1, 2, 3$ are rational.

Suppose, towards a contradiction, that there is no solution to the system comprised by (2) and (3). Then, by the argument in the proof of Lemma 5 there is no solution to System $S1$. Lemma 3 with $\mathbf{F} = \mathbf{R}$ implies that there is a real vector (θ, η, π) such that

$$\theta \cdot A + \eta \cdot B + \pi \cdot E = 0 \text{ and } \eta \geq 0, \pi > 0.$$

Recall that $B_4 = 0$ and $E_4 = 1$, so we obtain that $\theta \cdot A_4 + \pi = 0$.

Let $(q^k)_{k=1}^K$ be vectors of prices such that the dataset $(x^k, q^k)_{k=1}^K$ satisfies SA-EDU and $\log q_t^k \in \mathbf{Q}$ for all k and s. (Such $(q^k)_{k=1}^K$ exists by Lemma 6.) Construct matrices A', B', and E' from this dataset in the same way as A, B, and E is constructed in the proof of Lemma 5. Note that only the prices are different in (x^k, q^k) compared to (x^k, p^k). So $E' = E$, $B' = B$ and $A'_i = A_i$ for $i = 1, 2, 3$. Since only prices q^k are different in this dataset, only A'_4 may be different from A_4.

By Lemma 6, we can choose prices q^k such that $|\theta \cdot A'_4 - \theta \cdot A_4| < \pi/2$. We have shown that $\theta \cdot A_4 = -\pi$, so the choice of prices q^k guarantees that $\theta \cdot A'_4 < 0$. Let $\pi' = -\theta \cdot A'_4 > 0$.

Note that $\theta \cdot A'_i + \eta \cdot B'_i + \pi' E_i = 0$ for $i = 1, 2, 3$, as (θ, η, π) solves system $S2$ for matrices A, B and E, and $A'_i = A_i$, $B'_i = B_i$ and $E_i = 0$ for $i = 1, 2, 3$. Finally, $B_4 = 0$ so

$$\theta \cdot A'_4 + \eta \cdot B'_4 + \pi' E_4 = \theta \cdot A'_4 + \pi' = 0.$$

We also have that $\eta \geq 0$ and $\pi' > 0$. Therefore θ, η, and π' constitute a solution $S2$ for matrices A', B', and E'.

Lemma 3 then implies that there is no solution to $S1$ for matrices A', B', and E'. So there is no solution to the system comprised by (2) and (3) in the proof of Lemma 5. However, this contradicts Lemma 5 because the data (x^k, q^k) satisfies SA-EDU and $\log q_t^k \in \mathbf{Q}$ for all $k = 1, \ldots K$ and $t = 1, \ldots, T$.

9 Proof of Theorem 1 (ii), (iii), and (iv)

In this section, we provide a proof of Theorem 1 (ii), (iii), and (iv). Since all the proofs are similar, we give a detailed proof for (iv) and then explain how the proof for (iv) are different for the proofs for (ii) and (iii), if any.

Lemma 8. *Let $(x^k, p^k)_{k=1}^K$ be a dataset. The following statements are equivalent:*

1. *$(x^k, p^k)_{k=1}^K$ is future biased QHD rational.*

2. *There are strictly positive numbers v_t^k, λ^k, $\beta \geq 1$, and $\delta \in (0, 1]$, for $t = 0, \ldots, T$ and $k = 1, \ldots, K$, such that*

$$\begin{aligned} v_t^k &= \lambda^k p_t^k \textit{ if } t = 0, \\ \beta\delta^t v_t^k &= \lambda^k p_t^k \textit{ if } t > 0, \\ x_t^k > x_{t'}^{k'} &\Rightarrow v_t^k \leq v_{t'}^{k'}. \end{aligned}$$

Proof. We shall prove that (1) implies (2). Let $(x^k, p^k)_{k=1}^K$ be future biased QHD rational. Let $\beta \geq 1$, $\delta \in (0, 1]$, and $u : \mathbf{R}_+ \to \mathbf{R}$ be as in the definition of future biased QHD rational data. Then (see, for example, Theorem 28.3 of Rockafellar (1997)), there are numbers $\lambda^k \geq 0$, $k = 1, \ldots, K$ such that if we let

$$v_t^k = \frac{\lambda^k p_t^k}{\beta\delta^t} \text{ if } t > 0; \quad v_t^k = \lambda^k p_t^k \text{ if } t = 0$$

then $v_t^k \in \partial u(x_t^k)$ if $x_t^k > 0$, and there is $\underline{w} \in \partial u(x_t^k)$ with $v_t^k \geq \underline{w}$ if $x_t^k = 0$. In fact, it is easy to see that $\lambda^k > 0$, and therefore $v_t^k > 0$.

By the concavity of u, and the consequent monotonicity of $\partial u(x_t^k)$ (see Theorem 24.8 of Rockafellar (1997)), if $x_t^k > x_{t'}^{k'} > 0$, $v_t^k \in \partial u(x_t^k)$, and $v_{t'}^{k'} \in \partial u(x_{t'}^{k'})$, then $v_t^k \leq v_{t'}^{k'}$. If $x_t^k > x_{t'}^{k'} = 0$, then $\underline{w} \in \partial u(x_{t'}^{k'})$ with $v_{t'}^{k'} \geq \underline{w}$. So $v_t^k \leq \underline{w} \leq v_{t'}^{k'}$.

In second place, we show that (2) implies (1). Suppose that the numbers v_t^k, λ^k, β, δ, for $t \in T$ and $k \in K$, are as in (2).

Enumerate the elements in $\mathcal{X}$ in increasing order:

$$y_1 < y_2 < \ldots < y_n$$

Let

$$\underline{y}_i = \min\{v_t^k : x_t^k = y_i\} \text{ and } \bar{y}_i = \max\{v_t^k : x_t^k = y_i\}.$$

Let $z_i = (y_i + y_{i+1})/2$, $i = 1, \ldots, n-1$; $z_0 = 0$, and $z_n = y_n + 1$. Let f be a correspondence defined as follows:

$$f(z) = \begin{cases} [\underline{y}_i, \bar{y}_i] & \text{if } z = y_i, \\ \max\{\bar{y}_i : z < y_i\} & \text{if } y_n > z \text{ and } \forall i (z \neq y_i), \\ \underline{y}_n/2 & \text{if } y_n < z. \end{cases}$$

By assumption of the numbers v_t^k, we have that, when $y < y'$, $v \in f(y)$ and $v' \in f(y')$, then $v \leq v'$. Then the correspondence f is monotone and there is a concave function u for which $\partial u = f$ (Theorem 24.8 of Rockafellar (1997)). Given that $v_t^k > 0$ all the elements in the range of f are positive, and therefore u is strictly increasing.

Finally, for all (k,t), $\lambda^k p_t^k/(\beta\delta^t) = v_t^k \in \partial u(v_t^k)$ if $t > 0$; $\lambda^k p_t^k = v_t^k \in \partial u(v_t^k)$ if $t = 0$ and therefore the first-order conditions to a maximum choice of x hold at x_t^k. Since u is concave the first-order conditions are sufficient. The data is therefore future biased QHD rational. □

9.1 Necessity

Lemma 9. *If a dataset $(x^k, p^k)_{k=1}^K$ is future biased QHD rational, then it satisfies SA-F-QHD..*

Proof. Let $(x^k, p^k)_{k=1}^K$ be future biased QHD rational, and let $\beta \geq 1$, $\delta \in (0,1]$, and $u : \mathbf{R}_+ \to \mathbf{R}$ be as in the definition of future biased QHD rational. By Lemma 8, there exists a strictly positive solution v_t^k, λ^k, β, δ to the system in Statement (2) of Lemma 8 with $v_t^k \in \partial u(x_t^k)$ when $x_t^k > 0$, and $v_t^k \geq \underline{w} \in \partial u(x_t^k)$ when $x_t^k = 0$. Moreover, $v_t^k = \lambda^k p_t^k / D(t)$, where

$$D(t) = \begin{cases} 1 & \text{if } t = 0, \\ \beta\delta^t & \text{if } t > 0. \end{cases}$$

Let $(x_{t_i}^{k_i}, x_{t_i'}^{k_i'})_{i=1}^n$ be a sequence satisfying the four conditions in SA-F-QHD. Then $x_{t_i}^{k_i} > x_{t_i'}^{k_i'}$. Suppose that $x_{t_i'}^{k_i'} > 0$. Then, $v_{t_i}^{k_i} \in \partial u(x_{t_i}^{k_i})$ and $v_{t_i'}^{k_i'} \in \partial u(x_{t_i'}^{k_i'})$. By the concavity of u, it follows that $v_{t_i}^{k_i} \leq v_{t_i'}^{k_i'}$ (see Theorem 24.8 of Rockafellar (1997)). Similarly, if $x_{t_i'}^{k_i'} = 0$, then $v_{t_i}^{k_i} \in \partial u(x_{t_i}^{k_i})$ and $v_{t_i'}^{k_i'} \geq \underline{w} \in \partial u(x_{t_i'}^{k_i'})$, so that $v_{t_i}^{k_i} \leq v_{t_i'}^{k_i'}$.

Therefore,

$$1 \geq \prod_{i=1}^n \frac{\lambda^{k_i} D(t_i') p_{t_i}^{k_i}}{\lambda^{k_i'} D(t_i) p_{t_i'}^{k_i'}} = \prod_{i=1}^n \frac{D(t_i') p_{t_i}^{k_i}}{D(t_i) p_{t_i'}^{k_i'}} = \frac{\beta^{\#\{i:t_i'>0\} - \#\{i:t_i>0\}}}{\delta^{(\sum t_i - \sum t_i')}} \prod_{i=1}^n \frac{p_{t_i}^{k_i}}{p_{t_i'}^{k_i'}} \geq \prod_{i=1}^n \frac{p_{t_i}^{k_i}}{p_{t_i'}^{k_i'}},$$

where the first equality holds by (4) of SA-F-QHD; and the numbers λ^k appear the same number of times in the denominator as in the numerator of this product. Moreover, the last inequality holds by (2) and (3) of SA-F-QHD; and hence $\sum t_i \geq \sum t_i'$, and $\#\{i : t_i' > 0\} \leq \#\{i : t_i > 0\}$. □

9.2 Sufficiency

Lemma 10. *Let data* $(x^k, p^k)_{k=1}^{k}$ *satisfy SA-F-QHD. Suppose that* $\log(p_t^k) \in \mathbf{Q}$ *for all* k *and* t*. Then there are numbers* v_t^k*,* λ^k*,* β*,* δ*, for* $t \in T$ *and* $k \in K$ *satisfying* (2) *in Lemma 8.*

Lemma 11. *Let data* $(x^k, p^k)_{k=1}^{k}$ *satisfy SA-F-QHD. Then for all positive numbers* $\overline{\varepsilon}$*, there exists* $q_t^k \in [p_t^k - \overline{\varepsilon}, p_t^k]$ *for all* $t \in T$ *and* $k \in K$ *such that* $\log q_t^k \in \mathbf{Q}$ *and the dataset* $(x^k, q^k)_{k=1}^{k}$ *satisfy SA-F-QHD.*

Lemma 12. *Let data* $(x^k, p^k)_{k=1}^{k}$ *satisfy SA-F-QHD. Then there are numbers* v_t^k*,* λ^k*,* β*,* δ*, for* $t \in T$ *and* $k \in K$ *satisfying* (2) *in Lemma 8.*

Lemma 11 and 12 hold as in the previous section.

9.3 Proof of Lemma 10

We linearize the equation in System (2) of Lemma 8. The result is:

$$\log v(x_t^k) - \log \lambda^k - \log p_t^k = 0 \text{ if } t = 0, \tag{5}$$

$$\log v(x_t^k) + \log \beta + t \log \delta - \log \lambda^k - \log p_t^k = 0 \text{ if } t > 0, \tag{6}$$

$$x > x' \Rightarrow \log v(x') \geq \log v(x), \tag{7}$$

$$\log \beta \geq 0, \tag{8}$$

$$\log \delta \leq 0. \tag{9}$$

In the system comprised by (5), (6) (7), (8) and (9), the unknowns are the real numbers $\log v_t^k$, $\log \delta$, $k = 1, \ldots, K$ and $t = 1, \ldots, T$.

First, we are going to write the system of inequalities (5), (6), and (7) in matrix form.

A system of linear inequalities

We shall define a matrix A such that there are positive numbers v_t^k, λ^k, β, δ the logs of which satisfy Equation (2) if and only if there is a solution $u \in \mathbf{R}^{K\times(T+1)+2+K+1}$ to the system of equations

$$A \cdot u = 0,$$

and for which the last component of u is strictly positive.

Let A be a matrix with $K \times (T+1)$ rows and $K \times (T+1) + 1 + K + 1$ columns, defined as follows: We have one row for every pair (k, t); one column for every pair (k, t); two columns for each k; and two additional columns. Organize the columns so that we first have the $K \times (T+1)$ columns for the pairs (k, t); then two columns, which we shall refer to as the β-column and δ-column, respectively; then K columns (one for each k); and finally one last column. In the row corresponding to (k,t) the matrix has zeroes everywhere with the following exceptions: it has a 1 in the column for (k,t); it has a 1 if $t>0$ and it has a 0 if $t=0$ in the β column; it has a t in the δ column; it has a -1 in the column for k; and $-\log p_t^k$ in the very last column.

Thus, matrix A looks as follows:

$$\begin{array}{r} \\ \vdots \\ (k,t{=}0) \\ (k,t'{>}0) \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccccccccc} (1,1) & \cdots & (k,t) & (k,t') & \cdots & (K,T) & \beta & \delta & 1 & \cdots & k & \cdots & K & p \end{array} \\ \left[\begin{array}{cccccc|c|c|ccccc|c} \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 & 0 & t & 0 & \cdots & -1 & \cdots & 0 & -\log p_t^k \\ 0 & \cdots & 0 & 1 & \cdots & 0 & 1 & t' & 0 & \cdots & -1 & \cdots & 0 & -\log p_{t'}^{k'} \\ \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots \end{array}\right] \end{array}$$

Consider the system $A\cdot u=0$. If there are numbers solving Equation (5), then these define a solution $u\in\mathbf{R}^{K\times(T+1)+2+K+1}$ for which the last component is 1. If, on the other hand, there is a solution $u\in\mathbf{R}^{K\times(T+1)+2+K+1}$ to the system $A\cdot u=0$ in which the last component is strictly positive, then by dividing through by the last component of u we obtain numbers that solve Equation (5).

In second place, we write the system of inequalities (7) in matrix form. Let B be a matrix B with $K\times(T+1)+1+K+1$ columns. Define B as follows: One row for every pair (k,t) and (k',t') with $x_t^k > x_{t'}^{k'}$; in the row corresponding to (k,t) and (k',t') we have zeroes everywhere with the exception of a -1 in the column for (k,t) and a 1 in the column for (k',t') . Finally, we have last two rows, where we have zero everywhere with one exception. In the first row, we have a 1 at $K\times(T+1)+1$th column; in the second row, we have a -1 at $K\times(T+1)+2$th column. We shall refer to the first last row as the β-row, which captures (8). We also shall refer to the second row as the δ-row, which captures (9).

(For present biased QHD, we have -1 instead of 1 in β column. For general QHD, we do not have β row.)

In third place, we have a matrix E that captures the requirement that the last component of a solution be strictly positive. The matrix E has a single row and $K\times(T+1)+2+K+1$ columns. It has zeroes everywhere except for 1 in the last column.

To sum up, there is a solution to system (5), (7), (8), and (9) if and only if there is a vector $u\in\mathbf{R}^{K\times(T+1)+2+K+1}$ that solves the system of equations and linear inequalities

$$S1:\begin{cases} A\cdot u=0, \\ B\cdot u\geq 0, \\ E\cdot u\gg 0. \end{cases}$$

Theorem of the Alternative

The entries of A, B, and E are integer numbers, with the exception of the last column of A. Under the hypothesis of the lemma we are proving, the last column consists of rational numbers.

By Lemma 4, then, there is such a solution u to $S1$ if and only if there is no vector (θ, η, π) that solves the system of equations and linear inequalities

$$S2: \begin{cases} \theta \cdot A + \eta \cdot B + \pi \cdot E = 0, \\ \eta \geq 0, \\ \pi > 0. \end{cases}$$

In the following, we shall prove that the non-existence of a solution u implies that the data must violate SA-EDU. Suppose then that there is no solution u and let (θ, η, π) be a rational vector as above, solving system $S2$.

By multiplying (θ, η, π) by any positive integer we obtain new vectors that solve $S2$, so we can take (θ, η, π) to be integer vectors.

Henceforth, we use the following notational convention: For a matrix D with $K \times (T + 1) + 2 + K + 1$ columns, write D_1 for the submatrix of D corresponding to the first $K \times (T + 1)$ columns; let D_2 be the submatrix corresponding to the following one column (i.e., β-column); let D_3 be the submatrix corresponding to the following one column (i.e., δ-column); D_4 correspond to the next K columns; and D_5 to the last column. Thus, $D = [D_1|D_2|D_3|D_4|\, D_5]$.

Claim 9. *(i)* $\theta \cdot A_1 + \eta \cdot B_1 = 0$*; (ii)* $\theta \cdot A_2 + \eta \cdot B_2 = 0$*; (iii)* $\theta \cdot A_3 + \eta \cdot B_3 = 0$*; (iv)* $\theta \cdot A_4 = 0$*; and (v)* $\theta \cdot A_5 + \pi \cdot E_5 = 0$.

Proof. Since $\theta \cdot A + \eta \cdot B + \pi \cdot E = 0$, then $\theta \cdot A_i + \eta \cdot B_i + \pi \cdot E_i = 0$ for all $i = 1, \ldots, 5$. Moreover, since B_4, B_5, E_1, E_2, E_3, and E_4 are zero matrices, we obtain the claim. $\square$

For convenience, we transform the matrices A and B using θ and η.

Transform the matrices A and B

Lets define a matrix A^* from A by letting A^* have the same number of columns as A and including

1. θ_r copies of the rth row when $\theta_r > 0$;
2. omitting row r when $\theta_r = 0$;

3. and θ_r copies of the rth row multiplied by -1 when $\theta_r < 0$.

We refer to rows that are copies of some r with $\theta_r > 0$ as *original* rows, and to those that are copies of some r with $\theta_r < 0$ as *converted* rows.

Similarly, we define the matrix B^* from B by including the same columns as B and η_r copies of each row (and thus omitting row r when $\eta_r = 0$; recall that $\eta_r \geq 0$ for all r).

Claim 10. *For any (k,t), all the entries in the column for (k,t) in A_1^* are of the same sign.*

Proof. By definition of A, the column for (k,t) will have zero in all its entries with the exception of the row for (k,t). In A^*, for each (k,t), there are three mutually exclusive possibilities: the row for (k,t) in A can (i) not appear in A^*, (ii) it can appear as original, or (iii) it can appear as converted. This shows the claim. □

Claim 11. *There exists a sequence of pairs $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^{n^*}$ that satisfies (1) in SA-EDU.*

Proof. We define such a sequence by induction. Let $B^1 = B^*$. Given B^i, define B^{i+1} as follows.

Denote by $>^i$ the binary relation on $\mathcal{X}$ defined by $z >^i z'$ if $z > z'$ and there is at least one pair (k,t) and (k',t') for which (i) $x_t^k > x_{t'}^{k'}$; (ii) $z = x_t^k$ and $z' = x_{t'}^{k'}$; and (iii) the row corresponding $x_t^k > x_{t'}^{k'}$ in B has strictly positive weight in η.

The binary relation $>^i$ cannot exhibit cycles because $>^i \subseteq >$. There is therefore at least one sequence $z_1^i, \ldots z_{L_i}^i$ in $\mathcal{X}$ such that $z_j^i >^i z_{j+1}^i$ for all $j = 1, \ldots, L_i - 1$ and with the property that there is no $z \in \mathcal{X}$ with $z >^i z_1^i$ or $z_{L_i}^i >^i z$.

Let the matrix B^{i+1} be defined as the matrix obtained from B^i by omitting one copy of the row corresponding to $z_j^i > z_{j+1}^i$, for all $j = 1, \ldots L_i - 1$.

The matrix B^{i+1} has strictly fewer rows than B^i. There is therefore n^* for which B^{n^*+1} either has no more rows, or $B_1^{n^*+1}$ has only zeroes in all its entries (its rows are copies of the δ-row which has only zeroes in its first $K \times (T+1)$ columns).

Define a sequence of pairs $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^{n^*}$ by letting $x_{t_i}^{k_i} = z_1^i$ and $x_{t'_i}^{k'_i} = z_{L_i}^i$. Note that, as a result, $x_{t_i}^{k_i} > x_{t'_i}^{k'_i}$ for all i. Therefore the sequence of pairs $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^{n^*}$ satisfies condition (1) in SA-EDU. □

We shall use the sequence of pairs $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^{n^*}$ as our candidate violation of SA-EDU.

Consider a sequence of matrices A^i, $i = 1, \ldots, n^*$ defined as follows. Let $A^1 = A^*$, $B^1 = B^*$, and

$$C^1 = \begin{bmatrix} A^1 \\ B^1 \end{bmatrix}.$$

Observe that the rows of C^1 add to the null vector by Claim 9.

We shall proceed by induction. Suppose that A^i has been defined, and that the rows of

$$C^i = \begin{bmatrix} A^i \\ B^i \end{bmatrix}$$

add to the null vector.

Recall the definition of the sequence

$$x_{t_i}^{k_i} = z_1^i > \ldots > z_{L_i}^i = x_{t_i'}^{k_i'}.$$

There is no $z \in \mathcal{X}$ with $z >^i z_1^i$ or $z_{L_i}^i >^i z$, so in order for the rows of C^i to add to zero there must be a -1 in A_1^i in the column corresponding to (k_i', t_i') and a 1 in A_1^i in the column corresponding to (k_i, t_i). Let r_i be a row in A^i corresponding to (k_i, t_i), and r_i' be a row corresponding to (k_i', t_i'). The existence of a -1 in A_1^i in the column corresponding to (k_i', t_i'), and a 1 in A_1^i in the column corresponding to (k_i, t_i), ensures that r_i and r_i' exist. Note that the row r_i' is a converted row while r_i is original. Let A^{i+1} be defined from A^i by deleting the two rows, r_i and r_i'.

Claim 12. *The sum of r_i, r_i', and the rows of B^i which are deleted when forming B^{i+1} (corresponding to the pairs $z_j^i > z_{j+1}^i$, $j = 1, \ldots, L_i - 1$) add to the null vector.*

Proof. Recall that $z_j^i >^i z_{j+1}^i$ for all $j = 1, \ldots, L_i - 1$. So when we add the rows corresponding to $z_j^i >^i z_{j+1}^i$ and $z_{j+1}^i >^i z_{j+2}^i$, then the entries in the column for (k, t) with $x_t^k = z_{j+1}^i$ cancel out and the sum is zero in that entry. Thus, when we add the rows of B^i that are not in B^{i+1} we obtain a vector that is 0 everywhere except the columns corresponding to z_1^i and $z_{L_i}^i$. This vector cancels out with $r_i + r_i'$, by definition of r_i and r_i'. $\square$

Claim 13. *The matrix A^* can be partitioned into pairs of rows as follows:*

$$A^* = \begin{bmatrix} r_1 \\ r_1' \\ \vdots \\ r_i \\ r_i' \\ \vdots \\ r_{n^*} \\ r_{n^*}', \end{bmatrix}$$

in which the rows r_i' are converted and the rows r_i are original.

Proof. For each i, A^{i+1} differs from A^i in that the rows r_i and r'_i are removed from A^i to form A^{i+1}. We shall prove that A^* is composed of the $2n^*$ rows r_i, r'_i.

First note that since the rows of C^i add up to the null vector, and A^{i+1} and B^{i+1} are obtained from A^i and B^i by removing a collection of rows that add up to zero, then the rows of C^{i+1} must add up to zero as well.

By way of contradiction, suppose that there exist rows left after removing r_{n^*} and r'_{n^*}. Then, by the argument above, the rows of the matrix C^{n^*+1} must add to the null vector. If there are rows left, then the matrix C^{n^*+1} is well defined.

By definition of the sequence B^i, however, B^{n^*+1} has all its entries equal to zero, or has no rows. Hence, the rows remaining in $A_1^{n^*+1}$ must add up to zero. By Claim 10, the entries of a column (k,t) of A^* are always of the same sign. Moreover, each row of A^* has a non-zero element in the first $K \times S$ columns. Therefore, no subset of the columns of A_1^* can sum to the null vector. $\square$

Claim 14. *(i) For any k and t, if $(k_i, t_i) = (k,t)$ for some i, then the row r_i corresponding to (k,t) appears as original in A^*. Similarly, if $(k'_i, t'_i) = (k', t')$ for some i, then the row corresponding to (k,t) appears converted in A^*.*
(ii) If the row corresponding to (k,t) appears as original in A^*, then there is some i with $(k_i, t_i) = (k,t)$. Similarly, if the row corresponding to (k,t) appears converted in A^*, then there is i with $(k'_i, t'_i) = (k,t)$.

Proof. (i) is true by definition of $(x^{k_i}_{t_i}, x^{k'_i}_{t'_i})$. (ii) is immediate from Claim 13 because if the row corresponding to (k,t) appears original in A^* then it equals r_i for some i, and then $x^k_t = x^{k_i}_{t_i}$. Similarly when the row appears converted. $\square$

Claim 15. *The sequence $(x^{k_i}_{t_i}, x^{k'_i}_{t'_i})_{i=1}^{n^*}$ satisfies (2), (3), and (4) in SA-EDU.*

Proof. We first establish (2). Note that A_3^* is a vector, and in row r the entry of A_3^* is as follows. There must be a (k,t) of which r is a copy. Then the component at row r of A_3^* is t if r is original and $-t$ if r is converted. Now, when r appears as original there is some i for which $t = t_i$, when r appears as converted there is some i for which $t = t'_i$. So for each r there is i such that $(A_3^*)_r$ is either t_i or $-t'_i$.

By Claim 9 (iii), $\theta \cdot A_3 + \eta \cdot B_3 = 0$. Recall that $\theta \cdot A_3$ equals the sum of the rows of A_3^*. Moreover, B_3 is a vector that has zeroes everywhere except a -1 in the δ row (i.e., $K \times (T+1) + 2$th row). Therefore, the sum of the rows of A_3^* equals $\eta_{K\times(T+1)+2}$, where

$\eta_{K\times(T+1)+2}$ is the $K\times(T+1)+2$th element of η. Since $\eta \geq 0$, therefore, $\sum_{i:t_i>0} t_i - \sum_{i:t'_i>0} t'_i = \eta_{K\times(T+1)+2} \geq 0$, and condition (2) in the axiom is satisfied.

Next, we show (3). By Claim 9 (ii), $\theta \cdot A_2 + \eta \cdot B_2 = 0$. Recall that $\theta \cdot A_2$ equals the sum of the rows of A_2^*. Moreover, B_2 is a vector that has zeroes everywhere except a 1 in the β row (i.e., $K \times (T+1) + 1$th row). Therefore, the sum of the rows of A_2^* equals $\eta_{K\times(T+1)+1}$, where $\eta_{K\times(T+1)+1}$ is the $K \times (T+1) + 1$th element of η. Since $\eta \geq 0$, therefore, $-\#\{i : t_i > 0\} + \#\{i : t'_i > 0\} = \eta_{K\times(T+1)+1} \geq 0$, and condition (3) in the axiom is satisfied.

(For present biased QHDU, B_2 is a vector that has zeroes everywhere except a -1 in the β row (i.e., $K \times (T+1) + 1$th row). Hence, $\#\{i : t_i > 0\} - \#\{i : t'_i > 0\} = \eta_{K\times(T+1)+1} \geq 0$, and condition (3) in present biased QHDU is satisfied. For general QHDU, B_2 is a zero vector in the β row (i.e., $K \times (T+1) + 1$th row). Hence, $\#\{i : t_i > 0\} - \#\{i : t'_i > 0\} = 0$, and condition (3) in general biased QHDU is satisfied.)

Now we turn to (4). By Claim 9 (iv), the rows of A_4^* add up to zero. Therefore, the number of times that k appears in an original row equals the number of times that it appears in a converted row. By Claim 14, then, the number of times k appears as k_i equals the number of times it appears as k'_i. Therefore condition (4) in the axiom is satisfied. □

Finally, in the following, we show that

$$\prod_{i=1}^{n^*} \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}} > 1,$$

which finishes the proof of Lemma 5 as the sequence $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^{n^*}$ would then exhibit a violation of SA-EDU.

Claim 16. $\prod_{i=1}^{n^*} \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}} > 1$.

Proof. By Claim 9 (iv) and the fact that the submatrix E_4 equals the scalar 1, we obtain

$$0 = \theta \cdot A_4 + \pi E_4 = (\sum_{i=1}^{n^*}(r_i + r'_i))_4 + \pi,$$

where $(\sum_{i=1}^{n^*}(r_i+r'_i))_4$ is the (scalar) sum of the entries of A_4^*. Recall that $-\log p_{t_i}^{k_i}$ is the last entry of row r_i and that $\log p_{t'_i}^{k'_i}$ is the last entry of row r'_i, as r'_i is converted and r_i original. Therefore the sum of the rows of A_4^* are $\sum_{i=1}^{n^*} \log(p_{t'_i}^{k'_i}/p_{t_i}^{k_i})$. Then,

$$\sum_{i=1}^{n^*} \log(p_{t'_i}^{k'_i}/p_{t_i}^{k_i}) = -\pi < 0.$$

Thus

$$\prod_{i=1}^{n^*} \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}} > 1.$$

□

9.4 GTD and MTD

The proof that GTD rational is equivalent to SA-GTD is identical to the result in Echenique and Saito (2013a). One simply needs to interpret states in Echenique and Saito (2013a) as time periods. The GTD model is then the same as the subjective expected utility model.

The proof that SA-MTD es equivalent to MTD rationality requires the following modification of he argument in Echenique and Saito (2013a).

To see that SA-MGTD is necessary, let $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^n$ under the conditions of the axiom. The first-order condition is $D(t)u'(x_t^k) = \lambda^k p_t$. Then

$$\begin{aligned} 1 \geq \prod_{i=1}^n \frac{u'(x_{t_i}^{k_i})}{u'(x_{t'_i}^{k'_i})} &= \prod_{i=1}^n \frac{\lambda^{k_i} D(t'_i) p_{t_i}^{k_i}}{\lambda^{k'_i} D(t_i) p_{t'_i}^{k'_i}} \\ &= \prod_{i=1}^n \frac{D(t'_i) p_{t_i}^{k_i}}{D(t_i) p_{t'_i}^{k'_i}} \\ &= \prod_{i=1}^n \frac{D(t'_i)}{D(t_i)} \prod_{i=1}^n \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}} \\ &= \prod_{i=1}^n \frac{D(t'_{\pi(i)})}{D(t_i)} \prod_{i=1}^n \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}}. \end{aligned}$$

Since $t_i \geq t'_{\pi(i)}$ and D is decreasing it follows that

$$\frac{D(t'_{\pi(i)})}{D(t_i)} \geq 1.$$

Therefore we must have that

$$\prod_{i=1}^n \frac{p_{t_i}^{k_i}}{p_{t'_i}^{k'_i}}.$$

To see that it is sufficient, consider the following. We need to add rows to B to reflect that $D(t') \geq D(t)$ when $t \geq t'$. In the solution to the dual, we follow the steps of the proof until we construct a regular sequence $(x_{t_i}^{k_i}, x_{t'_i}^{k'_i})_{i=1}^n$. Such a sequence corresponds to a decomposition of A^* into pairs of rows $(r_i, r'_i)_{i=1}^n$ in which r_i is original and r'_i is reversed.

Now consider the row corresponding to t. The rows of B are no longer all zero in the column for B. The sum of the rows of $A^* + B^*$ equal zero. As usual we can eliminate pairs of rows of B such that $1_{t'} - 1_t + 1_t - 1_{t''} = 1_{t'} - 1_{t''}$. So in the matrix B^* all the entries in the column for t will be of the same sign. Let us say that they are all -11. Since the rows of $A^* + B^*$ is zero, the number of times that t appears in an original row minus the number of times that t appears in a reversed row equals the number of rows in B^* in which t has a -1. For each such row ρ of B^* there is some $t(\rho)$ in which B^* has a 1, so $t(\rho) \leq t$. We have assumed that there are only -1 in B^* in t so the number of reversed rows must be fewer than the number of original rows in which t is a part. For each such reversed row r_i' let $\pi(i)$ be one (distinct) original row in which t appears. Thus $t_i = t_\pi$ for such i. This exhausts all the reversed rows, and there may be some original rows that are not the image of π.

For each of these original rows r_i, there is a corresponding ρ_i in B^* with an entry of -1 in column t. Let r_j' be a reversed row in whys $t(\rho_i)$ appears (reversed rows must here exceed original rows, since the entries in B^* column $t(\rho_i)$ are positive). Let $\pi(i) = j$. Then $t_i \geq t(\rho_i) = t_\pi$.

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